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SEQUENCES OF HOMEOMORPHISMS ON THE n-SPHERE

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1. Introduction. In his recent book of mathematical problems, S. Ulam (see [1, p. 46]) states the following question as one which he and Borsuk have considered:

Given an arbitrary closed subset C of an n-sphere S, $n>0$, does there exist a sequence H_1, H_2, H_3, $\cdots$ of homeomorphisms of S onto itself such that for every p of S, $\lim_{k\to\infty} H_k(p)$ exists and is in C, and every point of C is such a limit?

This problem also occurs in the original *Scottish Book*, along with the remark that Borsuk has solved the problem for the case in which S is two-dimensional.

In this note an affirmative answer is obtained for the above question in the general case. The proof leans heavily upon a result which the author obtained in [2].

2. Admissible polyhedra. Let Σ be the set of all closed n-cubes which are contained in the euclidean n-space R^n and whose edges are parallel to the coordinate axes.

If $J\in\Sigma$, a subset A of the boundary of J is an α-set of J if A is the union of a collection of $(n-1)$-dimensional faces of J and for some such face σ, σ is contained in A while the $(n-1)$-dimensional face opposite σ is not contained in A.

A polyhedron P is *admissible* if there exists a sequence $P_1, \cdots, P_k$ of polyhedra such that: $P_1\in\Sigma$, $P_k=P$, and for each $i=1, \cdots, k-1$, $P_{i+1}=P_i\cup J_i$ where $J_i\in\Sigma$ and $P_i\cap J_i$ is an α-set of J_i.

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LEMMA 1. *If P is an admissible polyhedron, $J\in\Sigma$, $P\cap J$ is an α-set of J, and U and V are open sets which contain J and P respectively, then there exists a homeomorphism h of R_n onto itself such that $h[P\cup J]\subset V$ and h is the identity on R^n-U.*

PROOF. There is an $(n-1)$-dimensional face σ of J such that $\sigma\subset P\cap J$ and $\sigma'\not\subset P\cap J$, where σ' is the face of J which is opposite σ. We may assume without loss of generality that $R^n=R\times R^{n-1}$ and that J is situated so that $\sigma\subset\{0\}\times R^{n-1}$ and $\sigma'\subset\{1\}\times R^{n-1}$. There exists $\epsilon>0$, a closed set M which is contained in the projection of σ onto R^{n-1}, and an open set $W\supset M$ in R^{n-1} such that $[\epsilon, 1]\times M\supset J-V$, $(0, 1+\epsilon)\times W$ is disjoint from P, and $(0, 1+\epsilon)\times W\subset U$.

By Urysohn's lemma, there exists a continuous function ϕ on R^{n-1} into $[\epsilon, 1]$ such that $\phi(x)=\epsilon$ for $x\in M$ and $\phi(x)=1$ for $x\in R^{n-1}-W$.

We now define h as follows: h is the identity on $R^n-(0, 1+\epsilon)\times W$, and if $x\in W$ then h maps the segment from $(0, x)$ to $(1, x)$ linearly onto the segment from $(0, x)$ to $(\phi(x), x)$ and h maps the segment from $(1, x)$ to $(1+\epsilon, x)$ linearly onto the segment from $(\phi(x), x)$ to $(1+\epsilon, x)$. It is easy to see that h has the desired properties.

LEMMA 2. *If G is a connected, open subset of R^n, $P\subset G$ is an admissible polyhedron, and $Q\subset G$ is a nonempty open set, then there exists a homeomorphism H of R^n onto R^n such that $H[P]\subset Q$ and H is the identity on R^n-G.*

PROOF. There exists a sequence $P_1, \cdots, P_k$ of admissible polyhedra such that $P_1\in\Sigma$, $P_k=P$, and for each $i=1, \cdots, k-1$, we have $P_{i+1}=P_i\cup J_i$ where $J_i\in\Sigma$ and $P_i\cap J_i$ is an α-set of J_i. It is obvious that there is a homeomorphism h_1 of R^n onto R^n such that h_1 is the identity outside G and $h_1[P_1]\subset Q$. Because of continuity of h_1, there is an open set $V_1\supset P_1$ such that $h_1[V_1]\subset Q$. By Lemma 1, there is a homeomorphism h_2 of R^n onto R^n such that h_2 is the identity outside G and $h_2[P_2]\subset V_1$, and by continuity of h_2 there is an open set $V_2\supset P_2$ such that $h_2[V_2]\subset V_1$. Continuing in this manner, we obtain open sets $V_i\supset P_i$ for each i, and homeomorphisms h_i of R^n onto R^n which are the identity outside G and which satisfy $h_{i+1}[V_{i+1}]\subset V_i$. The composite homeomorphism $H=h_1h_2 \cdots h_k$ has the desired properties.

3. The main result.

THEOREM. *Let S be an n-sphere, $n>0$, and let C be a nonempty closed subset of S. Then there exists a sequence $H_1, H_2, H_3, \cdots$ of homeomor-*

phisms of S onto itself such that $H_k(p)=p$ for each $p\in C$ and each positive integer k, and $\lim_{k\to\infty} H_k(p)$ exists and is in C for each $p\in S-C$.

PROOF. Choose a point $q\in C$ and let Φ be a homeomorphism of $S-\{q\}$ onto euclidean n-space R^n. We define $C^*=\Phi[C-\{q\}]$. We are going to define a sequence $h_1, h_2, h_3, \cdots$ of homeomorphisms of R^n onto itself.

We enumerate the components of R^n-C^* in a sequence $G_1, G_2, G_3, \cdots$ and choose a point p_k in the boundary of G_k for each positive integer k. (We assume that R^n-C^* has an infinite number of components, since the case in which R^n-C^* has a finite number of components can be handled in a similar manner.)

It follows from the theorem proved in [2] that there exist sets $P(k, j)$ for k and j positive integers such that: each set $P(k, j)$ is a closed topological n-cell, $P(k, 1)\subset P(k, 2)\subset P(k, 3)\subset \cdots$ for each k, and $G_k=\bigcup_{j=1}^{\infty} P(k, j)$ for each k. It follows from the construction used in [2] that the sets $P(k, j)$ may be chosen so as to be admissible polyhedra, and we assume that this has been done. We choose non-empty open sets $Q(k, j)$ for all positive integers k and j so that $Q(k, j)\subset G_k$ and $\lim_{j\to\infty} Q(k, j)=\{p_k\}$. It is now easy to use Lemma 2 to construct for each positive integer k a homeomorphism h_k of R^n onto itself such that: $h_k[P(i, k)]\subset Q(i, k)$ for $1\leqq i\leqq k$, and h_k is the identity on $R^n-\bigcup_{i=1}^{k} G_i$. It is easy to see that $h_j(x)=x$ for all $x\in C^*$ and all j, and that $\lim_{j\to\infty} h_j(x)=p_k\in C^*$ for $x\in G_k$.

The desired homeomorphisms of S onto S are now obtained by defining

$$H_k(p) = \begin{cases} q & \text{for } p = q, \\ \Phi^{-1}h_k\Phi(p) & \text{for } p \in S - \{q\}. \end{cases}$$

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