ON AUTOMORPHISM GROUPS OF HOMOGENEOUS COMPLEX MANIFOLDS

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The purpose of this short note is to prove the following:

Theorem. Let $M$ be a homogeneous complex manifold of complex dimension $n$ and $G$ a transitive group of holomorphic transformations of $M$. Let $F$ be the complex Hilbert space of square integrable holomorphic $n$-forms on $M$. Then the natural unitary representation of $G$ in $F$ is irreducible, i.e., no proper closed subspace of $F$ is invariant by $G$.

The reproducing property of the Bergman kernel function [1] plays an essential role in our proof. As we consider not only bounded domains in $\mathbb{C}^n$, but also general complex manifolds, we replace the kernel function by the kernel form (see [2] for the definition and properties of the kernel form). If $M$ is a domain in $\mathbb{C}^n$, then $F$ can be identified with the space $F^*$ of square integrable holomorphic functions (with respect to the Euclidean measure of $\mathbb{C}^n$). For our purpose, it is, however, desirable to use $F$ even for a domain in $\mathbb{C}^n$, because every holomorphic transformation of $M$ induces, in a natural way, a unitary transformation of $F$.

We recall that, by a square integrable holomorphic $n$-form $f$, we mean

$$[(-1)^{1/2}]^{n^2} \int_M f \wedge \bar{f} < \infty.$$ 

The inner product in $F$ is defined by

$$(f, g) = [(-1)^{1/2}]^{n^2} \int_M f \wedge \bar{g}.$$ 

In general, $F$ may or may not be of infinite dimension, or may even be trivial.

Proof. If $F$ is trivial, that is, $F = \{0\}$, then the theorem is trivially true. We assume therefore that $F$ is nontrivial. Let $\{h_0, h_1, h_2, \ldots\}$ be a complete orthonormal basis for $F$. Then the kernel form

$$K(z, \bar{w}) = \sum h_j(z) \wedge \{h_j(w)\}^{-}$$

is a holomorphic $2n$-form on $M \times \overline{M}$, where $\overline{M}$ denotes the complex manifold whose complex structure is conjugate to that of $M$. As in

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the case of the kernel function, $K(z, w)$ is independent of choice of the basis chosen. By identifying $M$ with the diagonal of $M \times \overline{M}$, we can consider $K(z, \overline{z})$ as a $2n$-form on $M$. From the nontriviality of $F$ and the homogeneity of $M$, it follows that, for every point $z$ of $M$, there exists an element $f$ of $F$ which does not vanish at $z$. This implies immediately that $K(z, \overline{z})$ is different from zero at every point $z$ of $M$.

Let $F'$ be a closed subspace of $F$ invariant by $G$ and $F''$ the orthogonal complement of $F'$ in $F$. Let $\{f_0, f_1, \cdots \}$ (respectively $\{g_0, g_1, \cdots \}$) be a complete orthonormal basis for $F'$ (respectively $F''$). Since the kernel form is independent of choice of basis for $F$, we have that

$$K(z, w) = \sum f_j(z) \wedge \{f_j(w)\}^- + \sum g_k(z) \wedge \{g_k(w)\}^-.$$

Set

$$K'(z, w) = \sum f_j(z) \wedge \{f_j(w)\}^-.$$

Since $F'$ is invariant by $G$, so is $K'$. As both $K(z, \overline{z})$ and $K'(z, \overline{z})$ are forms of maximum degree on $M$ invariant by $G$ and $G$ is transitive on $M$, we have that

$$K'(z, \overline{z}) = c \cdot K(z, \overline{z}) \text{ for } z \in M,$$

where $c$ is a positive constant (provided that $F' \neq \{0\}$). Since both $K(z, w)$ and $K'(z, w)$ are holomorphic on $M \times \overline{M}$, we have that

$$K'(z, w) = c \cdot K(z, w) \text{ for } (z, w) \in M \times \overline{M}.$$

Take any $f''$ from $F''$. Then

$$\int_{[z] \times \overline{M}} K'(z, w) \wedge f''(w) = \sum f_j(z) \cdot \int_M \{f_j(w)\}^- \wedge f''(w) = 0$$

because

$$(f_j, f'') = 0.$$

On the other hand, we have

$$\int_{[z] \times \overline{M}} K'(z, w) \wedge f''(w) = \int_{[z] \times \overline{M}} c \cdot K(z, w) \wedge f''(w) = [(-1)^{1/2}]^n c \cdot f''(z)$$

because of the reproducing property of $K$:

$$\int_{[z] \times \overline{M}} K(z, w) \wedge f(w) = f(z) \text{ for every } f \in F.$$

Since $c \neq 0$, $f''$ must identically vanish. Hence, $F'' = \{0\}$. Q.E.D.
SEQUENCES OF HOMEOMORPHISMS ON THE n-SPHERE

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1. Introduction. In his recent book of mathematical problems, S. Ulam (see [1, p. 46]) states the following question as one which he and Borsuk have considered:

Given an arbitrary closed subset $C$ of an $n$-sphere $S$, $n > 0$, does there exist a sequence $H_1, H_2, H_3, \cdots$ of homeomorphisms of $S$ onto itself such that for every $p$ of $S$, $\lim_{k \to \infty} H_k(p)$ exists and is in $C$, and every point of $C$ is such a limit?

This problem also occurs in the original Scottish Book, along with the remark that Borsuk has solved the problem for the case in which $S$ is two-dimensional.

In this note an affirmative answer is obtained for the above question in the general case. The proof leans heavily upon a result which the author obtained in [2].

2. Admissible polyhedra. Let $\Sigma$ be the set of all closed $n$-cubes which are contained in the euclidean $n$-space $R^n$ and whose edges are parallel to the coordinate axes.

If $J \in \Sigma$, a subset $A$ of the boundary of $J$ is an $\alpha$-set of $J$ if $A$ is the union of a collection of $(n-1)$-dimensional faces of $J$ and for some such face $\sigma$, $\sigma$ is contained in $A$ while the $(n-1)$-dimensional face opposite $\sigma$ is not contained in $A$.

A polyhedron $P$ is admissible if there exists a sequence $P_1, \cdots, P_k$ of polyhedra such that: $P_1 \in \Sigma$, $P_k = P$, and for each $i = 1, \cdots, k-1$, $P_{i+1} = P_i \cup J_i$ where $J_i \in \Sigma$ and $P_i \cap J_i$ is an $\alpha$-set of $J_i$.

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