ON AUTOMORPHISM GROUPS OF HOMOGENEOUS COMPLEX MANIFOLDS

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The purpose of this short note is to prove the following:

THEOREM. *Let M be a homogeneous complex manifold of complex dimension n and G a transitive group of holomorphic transformations of M. Let F be the complex Hilbert space of square integrable holomorphic n-forms on M. Then the natural unitary representation of G in F is irreducible, i.e., no proper closed subspace of F is invariant by G.*

The reproducing property of the Bergman kernel function [1] plays an essential role in our proof. As we consider not only bounded domains in C^n, but also general complex manifolds, we replace the kernel function by the kernel form (see [2] for the definition and properties of the kernel form). If M is a domain in C^n, then F can be identified with the space F^* of square integrable holomorphic functions (with respect to the Euclidean measure of C^n). For our purpose, it is, however, desirable to use F even for a domain in C^n, because every holomorphic transformation of M induces, in a natural way, a unitary transformation of F.

We recall that, by a *square integrable* holomorphic n-form f, we mean

$$[(-1)^{1/2}]^{n^2}\int_M f\wedge\bar{f} < \infty.$$

The inner product in F is defined by

$$(f, g) = [(-1)^{1/2}]^{n^2}\int_M f\wedge\bar{g}.$$

In general, F may or may not be of infinite dimension, or may even be trivial.

PROOF. If F is trivial, that is, $F=\{0\}$, then the theorem is trivially true. We assume therefore that F is nontrivial. Let $\{h_0, h_1, h_2, \cdots\}$ be a complete orthonormal basis for F. Then the kernel form

$$K(z, \bar{w}) = \sum h_j(z)\wedge\{h_j(w)\}^-$$

is a holomorphic $2n$-form on $M\times\overline{M}$, where $\overline{M}$ denotes the complex manifold whose complex structure is conjugate to that of M. As in

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the case of the kernel function, $K(z, \bar{w})$ is independent of choice of the basis chosen. By identifying M with the diagonal of $M\times\overline{M}$, we can consider $K(z, \bar{z})$ as a $2n$-form on M. From the nontriviality of F and the homogeneity of M, it follows that, for every point z of M, there exists an element f of F which does not vanish at z. This implies immediately that $K(z, \bar{z})$ is different from zero at every point z of M.

Let F' be a closed subspace of F invariant by G and F'' the orthogonal complement of F' in F. Let $\{f_0, f_1, \cdots\}$ (respectively $\{g_0, g_1, \cdots\}$) be a complete orthonormal basis for F' (respectively F''). Since the kernel form is independent of choice of basis for F, we have that

$$K(z, \bar{w}) = \sum f_j(z) \wedge \{f_j(w)\}^- + \sum g_k(z) \wedge \{g_k(w)\}^-.$$

Set

$$K'(z, \bar{w}) = \sum f_j(z) \wedge \{f_j(w)\}^-.$$

Since F' is invariant by G, so is K'. As both $K(z, \bar{z})$ and $K'(z, \bar{z})$ are forms of maximum degree on M invariant by G and G is transitive on M, we have that

$$K'(z, \bar{z}) = c\cdot K(z, \bar{z}) \qquad \text{for } z \in M,$$

where c is a positive constant (provided that $F'\neq\{0\}$). Since both $K(z, \bar{w})$ and $K'(z, \bar{w})$ are holomorphic on $M\times\overline{M}$, we have that

$$K'(z, \bar{w}) = c\cdot K(z, \bar{w}) \qquad \text{for } (z, \bar{w}) \in M \times \overline{M}.$$

Take any f'' from F''. Then

$$\int_{\{z\}\times\overline{M}} K'(z, \bar{w}) \wedge f''(w) = \sum f_j(z)\cdot \int_{\overline{M}} \{f_j(w)\}^- \wedge f''(w) = 0$$

because

$$(f_j, f'') = 0.$$

On the other hand, we have

$$\int_{\{z\}\times\overline{M}} K'(z, \bar{w}) \wedge f''(w) = \int_{\{z\}\times\overline{M}} c\cdot K(z, \bar{w}) \wedge f''(w) = [(-1)^{1/2}]^{n^2} c\cdot f''(z)$$

because of the reproducing property of K:

$$\int_{\{z\}\times\overline{M}} K(z, \bar{w}) \wedge f(w) = f(z) \qquad \text{for every } f \in F.$$

Since $c\neq 0$, f'' must identically vanish. Hence, $F''=\{0\}$. Q.E.D.

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SEQUENCES OF HOMEOMORPHISMS ON THE n-SPHERE

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1. Introduction. In his recent book of mathematical problems, S. Ulam (see [1, p. 46]) states the following question as one which he and Borsuk have considered:

Given an arbitrary closed subset C of an n-sphere S, $n>0$, does there exist a sequence $H_1, H_2, H_3, \cdots$ of homeomorphisms of S onto itself such that for every p of S, $\lim_{k\to\infty} H_k(p)$ exists and is in C, and every point of C is such a limit?

This problem also occurs in the original *Scottish Book*, along with the remark that Borsuk has solved the problem for the case in which S is two-dimensional.

In this note an affirmative answer is obtained for the above question in the general case. The proof leans heavily upon a result which the author obtained in [2].

2. Admissible polyhedra. Let Σ be the set of all closed n-cubes which are contained in the euclidean n-space R^n and whose edges are parallel to the coordinate axes.

If $J\in\Sigma$, a subset A of the boundary of J is an α-set of J if A is the union of a collection of $(n-1)$-dimensional faces of J and for some such face σ, σ is contained in A while the $(n-1)$-dimensional face opposite σ is not contained in A.

A polyhedron P is *admissible* if there exists a sequence $P_1, \cdots, P_k$ of polyhedra such that: $P_1\in\Sigma$, $P_k=P$, and for each $i=1, \cdots, k-1$, $P_{i+1}=P_i\cup J_i$ where $J_i\in\Sigma$ and $P_i\cap J_i$ is an α-set of J_i.

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[1] The author is an Alfred P. Sloan Research Fellow.