ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A FUNCTIONAL EQUATION[1]

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We consider the question as to whether or not the hypothesis that

$$\lim_{t \to \infty} (f(t+s) - f(t)) = \lambda s \tag{1}$$

for all s implies the conclusion that

$$\lim_{t \to \infty} \frac{f(t)}{t} = \lambda \tag{2}$$

in the presence of appropriate auxiliary hypotheses. We show that the result is not valid without some additional hypothesis; and that either measurability of f or boundedness of f on an appropriate set of intervals is adequate.

In the presence of auxiliary hypotheses, the requirement that (1) hold for all s is apt to be unnecessarily strong. Theorem 1 covers a situation in which (1) is assumed for some nonzero s; while Theorems 2 and 3 cover situations, by reduction to Theorem 1, in which (1) is assumed for a more plentiful set but in which weaker side conditions are assumed.

Functions satisfying (1) arise in renewal theory [1]. In such cases, the functions usually satisfy stronger additional conditions than any we consider here, and (2) may often be obtained from such conditions instead of (1).

In what follows we assume that f is a real-valued function defined on a set containing a right ray and λ is a real number. It is clear that if (1) holds for s in some set S, then (1) holds for s in the additive subgroup of the real numbers generated by S; and we shall only consider such subgroups in our hypotheses.

THEOREM 1. *Let f be bounded on the finite subintervals of some right ray, and let* (1) *hold for s in the subgroup S, where $S \neq \{0\}$. Then* (2) *holds.*

PROOF. Let ϵ be positive. Take s as a positive element of S. Then there exists a positive t_1 such that

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$$|f(t+s)-f(t)-\lambda s| < \epsilon, \qquad (t \geqq t_1). \tag{3}$$

We further restrict t_1 so that f is bounded on finite subintervals of $[t_1, \infty)$. With brackets denoting integral part, we have

$$t = [(t-t_1)/s]s + r,$$

where $r \in [t_1, t_1+s)$. Let M be a bound for $|f|$ on $[t_1, t_1+s)$.

Then, for $t \geqq t_1$, using (3) and the fact that $r \geqq t_1$,

$$\left|\frac{f(t)}{t} - \lambda\right|$$

$$= \left| \frac{\sum_{n=1}^{[(t-t_1)/s]} (f(ns+r) - f((n-1)s+r)) + f(r) - \sum_{n=1}^{[(t-t_1)/s]} \lambda s - \lambda r}{t} \right|$$

$$= \left| \frac{\sum_{n=1}^{[(t-t_1)/s]} (f(ns+r) - f((n-1)s+r) - \lambda s) + f(r) - \lambda r}{t} \right|$$

$$\leqq \frac{\sum_{n=1}^{[(t-t_1)/s]} |f((n-1)s+r+s) - f((n-1)s+r) - \lambda s| + |f(r)| + |\lambda| r}{t}$$

$$\leqq \frac{[(t-t_1)/s]\epsilon + M + |\lambda|(t_1+s)}{t} \leqq \frac{\epsilon}{s} + \frac{M + |\lambda|(t_1+s)}{t}.$$

Hence

$$\limsup_{t\to\infty} \left|\frac{f(t)}{t} - \lambda\right| \leqq \frac{\epsilon}{s}.$$

Since ϵ was an arbitrary positive number, (2) holds.

THEOREM 2. *Let f be such that, given t_1, there exists a finite closed interval contained in $[t_1, \infty)$ on which f is bounded. Let* (1) *hold for s in the nondiscrete subgroup S. Then f is bounded on the finite intervals of some right ray, and* (2) *holds.*

PROOF. Let s_1 be a positive element of S. Choose t_1 so that $|f(t+s_1)-f(t)-\lambda s_1| < 1$ for $t \geqq t_1$. Choose a finite closed interval I_1 contained in $[t_1, \infty)$ on which f is bounded. Then f is also bounded on every translation of I_1 by a positive integral multiple of s_1.

Since S is a nondiscrete group, we can choose s_2 in S such that s_2 is positive and not greater than the length of I_1. Choose t_2 so that

$|f(t+s_2)-f(t)-\lambda s_2|<1$ for $t \geqq t_2$. Let I_2 be a translation of I_1 by a positive integral multiple of s_1 such that $I_2 \subset [t_2, \infty)$. Then f is bounded on every translation of I_2 by a non-negative integral multiple of s_2. However, since s_2 is not greater than the length of I_2; such translations cover some right ray, and f is bounded on the finite subintervals of this ray. Since S is not discrete, $S \neq \{0\}$; and, by Theorem 1, (2) holds.

Before proving Theorem 3, which considers the case that f is measurable, we prove several lemmas. In the proofs, in connection with words such as "measurability," "measure," and so forth, the related measure will be obvious.

LEMMA. *Let ν be Lebesgue outer measure for the real plane, let A be a set such that $\nu(A)>0$, and let α be less than 1. Then, for specified rectangular axes, there exists a square W with sides parallel to the axes such that $\nu(A \cap W)>\alpha\nu(W)$.*

PROOF. With simple modifications the proof of [3, Theorem A, p. 68] carries through.

LEMMA. *If μ is Lebesgue outer measure for the real line, A and B are sets of real numbers not of μ-measure zero, and $\alpha<1$; then there exist intervals U and V of equal length such that $\mu(A \cap U)>\alpha\mu(U)$ and $\mu(B \cap V)>\alpha\mu(V)$.*

PROOF. Let ν be Lebesgue outer measure for the real plane. Then $\nu(A \times B)=\mu(A)\mu(B)>0$. (Confer [3, p. 150, (9)]). Hence, by the above lemma, there exists a square W, which can be considered to be $U \times V$ where U and V are intervals of equal length, such that

$$\nu((A \times B) \cap W) > \alpha\nu(W) = \alpha\mu(U)\mu(V).$$

Then

$$\begin{aligned}\alpha\mu(U)\mu(V) &< \nu((A \times B) \cap W) = \nu((A \cap U) \times (B \cap V)) \\ &\leqq \nu((A \cap U) \times V) = \mu(A \cap U)\mu(V),\end{aligned}$$

and $\mu(A \cap U)>\alpha\mu(U)$. Similarly $\mu(B \cap V)>\mu(V)$.

LEMMA. *Let μ be Lebesgue outer measure for the real line. Let A and B be sets of real numbers, such that A is μ-measurable and neither set has μ-measure zero. Then the set $\{a-b \mid a \in A,\ b \in B\}$ contains an interval.*

PROOF. Once the last lemma is established, the proof follows closely a proof for the case that $B=A$.

Choose U and V as in the last lemma for $\alpha=3/4$. Let u_0 and v_0 be the midpoints of U and V, respectively, and let $l=\mu(U)=\mu(V)>0$.

Consider the set $((A\cap U)\oplus v_0)\cup((B\cap V)\oplus(u_0+x))$ where $|x| < l/2$. ($C\oplus D=\{r|c\in C, d\in D, r=c+d\}$, and similarly for other such notations). It is readily verified that this set is contained in the interval, $[u_0+v_0-l/2+\min(0, x), u_0+v_0+l/2+\max(0, x)]$, which has length $(l+|x|)$. Now $\mu((A\cap U)\oplus v_0)$ and $\mu((B\cap V)\oplus(u_0+x))$ are greater than $(3/4)l$, and $(A\cap U)\oplus v_0$ is measurable; so that if the sets were disjoint, the measure of the union would be greater than $(3/2)l$, and therefore greater than $(l+|x|)$. To avoid this contradiction we must have that the two sets intersect, so there exist a in A and b in B such that

$$a + v_0 = b + u_0 + x,$$

$$a - b = u_0 - v_0 + x.$$

Thus the set $\{a - b|a \in A, \ b \in B\}$ contains the interval $(u_0-v_0-l/2, u_0-v_0+l/2)$.

THEOREM 3. *Let μ be Lebesgue outer measure for the real line. Let f be such that, given t_1, there exists a μ-measurable subset of positive μ-measure contained in $[t_1, \infty)$ on which f is μ-measurable. Let* (1) *hold for s in a subgroup S with $\mu(S)>0$. Then, given t_1, there exists a finite closed interval contained in $[t_1, \infty)$ on which f is bounded, and* (2) *holds.*

PROOF. Let $S'=\{s|s\in S, 0<s<1\}$. Of course $\mu(S')>0$. For n a positive integer, let $S_n=\{s|s\in S', \ t\geqq n\Rightarrow|f(t+s)-f(t)-\lambda s|<1\}$. Now $S'=\bigcup_{n=1}^\infty S_n$, so $\mu(S')\leqq\sum_{n=1}^\infty\mu(S_n)$. Hence, we can take n so that $\mu(S_n)>0$.

Now, given $t_1\geqq n$, let A_0 be a measurable set of positive measure contained in $[t_1, \infty)$ on which f is measurable. For m a positive integer the set A_m, $A_m=\{t|t\in A_0, |f(t)|\leqq m\}$, is measurable and f is bounded on it. Moreover, $A_0=\bigcup_{m=1}^\infty A_m$, so some A_m has positive measure. Hence there is a measurable set A, of positive measure and contained in $[t_1, \infty)$, on which f is bounded. Furthermore, $A\oplus S_n\subset[t_1, \infty)$, and $A\oplus S_n=A\ominus(\ominus S_n)$. Thus $A\oplus S_n$ contains an interval by the last lemma. Finally, for t in A and s in S_n, we have $f(t)$, s, and $|f(t+s)-f(t)-\lambda s|$ bounded. Hence $f(t+s)$ is bounded for t in A and s in S_n, and f is bounded on an interval contained in $[t_1, \infty)$.

Since S has positive measure, it is nondenumerable and, therefore, nondiscrete. By Theorem 2, (2) holds.

We close with an example showing that some additional hypothesis is necessary to insure that the assumption of (1) for all s implies (2).

THEOREM 4. *There exists a function such that* (1) *holds for all s and* (2) *does not hold.*

PROOF. Let L be a real-valued additive function on the real numbers other than multiplication by a real number [2, Chapter III, §9, pp. 116–117].

Let f be defined by

$$f(t) = (t + |L(t)|)^{1/2}, \qquad (t \geqq 1).$$

Now, for t and $t+s$ greater than 1,

$$\begin{aligned} |f(t+s) - f(t)| &= |(t+s+|L(t+s)|)^{1/2} - (t+|L(t)|)^{1/2}| \\ &= \frac{|s + |L(t+s)| - |L(t)||}{(t+s+|L(t+s)|)^{1/2} + (t+|L(t)|)^{1/2}} \\ &= \frac{|s + |L(t)+L(s)| - |L(t)||}{(t+s+|L(t+s)|)^{1/2} + (t+|L(t)|)^{1/2}} \\ &\leqq \frac{|s| + |L(s)|}{(t)^{1/2}}. \end{aligned}$$

Hence $\lim_{t\to\infty} |f(t+s)-f(t)| = 0 \cdot s$ for all s.

However, $L(t)$ is unbounded in every interval, [2, pp. 116–117]; so $f(t)/t$ must be unbounded in every interval. Thus $f(t)/t$ does not have limit zero as $t \to \infty$, and (2) does not hold.

REFERENCES

1. D. Blackwell, *A renewal theorem*, Duke Math. J. vol. 15 (1948) pp. 145–150.

2. H. Hahn and A. Rosenthal, *Set functions*, Albuquerque, University of New Mexico Press, 1948.

3. P. R. Halmos, *Measure theory*, New York, Van Nostrand, 1950.

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