NOTE ON DEGREES OF PARTIAL FUNCTIONS

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The notion of one function's being recursive in another is normally considered only for full functions; but Davis [1, p. 171] has given a definition applicable also to partial functions. For one-argument functions (to which we restrict ourselves for the sake of simplicity) this reads: f is partial recursive in g if there is a completely computable functional $\Phi$ for which $f(x) = \Phi(g, x)$. And here $\Phi$ is called completely computable if for some partial recursive $h$ we have

$$\Phi(g, x) = t \iff (\exists y)(y^{(1)} \subseteq g \text{ and } h(x, y) = t),$$

$\{i^{(1)}\}$ being an effective enumeration of finite functions. In this note we shall argue that Davis' definition does not do justice to our intuitive idea of relative computability; we shall suggest an alternative definition; and we shall show that on either definition there are degrees which are not degrees of any full function.\footnote{This fact was known to Lacombe and Shoenfield as early as 1958; but the use of a category argument to establish it, and the consequent strengthening of the result, is new.} In fact we shall show that in a suitable sense "almost no" degrees are degrees of full functions.\footnote{We say that almost no semicharacteristic functions (see below) have a certain property, if the class of all sets whose semicharacteristic functions have this property is of first category; and that almost no (weak) degrees have a certain property if almost no semicharacteristic functions belong to (weak) degrees having that property. The latter usage is justified by the fact that every weak degree $\mathfrak{D}$ contains a semicharacteristic function (for example $c_\alpha^g$ where $\alpha = \{2^{x^\gamma} \mid f(x) \text{ is defined}\}$ for any $f \in \mathfrak{D}$. (Not every strong degree contains a semicharacteristic function; for example this is not true, by the argument of the following paragraph, for any degree of a full non-recursive function.) Observe finally that there exist functions (for example the function $g_0$ of the following paragraph) which are not of the same degree as any full function on Davis' definition, though they are on ours.}

Let $\gamma$ be any nonrecursive set, and let $g_0(2x) = 0$ for $x \in \gamma$, $g_0(2x + 1) = 0$ for $x \in \gamma$, $g_0(x)$ undefined otherwise. Let $c_\gamma$ be the characteristic function of $\gamma$. Then $c_\gamma(x)$ is 0 if $g_0(2x)$ is 0 and 1 if $g_0(2x + 1)$ is 0, so clearly $c$ is effectively computable from $g_0$ in the intuitive sense. None the less, since $g_0$ is a restriction of the constant function $n(x) = 0$,
$\lambda x \Phi(g_0, x)$ is a restriction of the partial recursive function $\lambda x \Phi(n, x)$ for every completely computable $\Phi$: but $c_\gamma$, being full and nonrecursive, cannot be a restriction of a partial recursive function. Hence $c$ is not partial recursive in $g_0$ in the sense of Davis.

We regard $f$ as effectively computable from $g$ in the intuitive sense if there exists a mechanical method by means of which every correct and no incorrect value of $f$ can be computed using only finitely many values of $g$. If we assume that this method can be formalized in some formal system with recursive rules of inference, we are led to the following amendment of Davis' definition.

$f$ is called partial recursive in $g$ if there is a recursively enumerable relation $R(x, y, t)$ for which

$$f(x) = t \iff (\exists y)(y^{[1]} \subseteq g \text{ and } R(x, y, t)).$$

Trivially, if $f$ is partial recursive in $g$ in Davis' sense, it is in ours too. For $g$ full the converse holds (cf. the Corollary of Theorem XIX in [2, p. 331]) but by the immediately preceding counterexample not for $g$ arbitrary. Two functions are called strongly (Turing) equivalent if each is partial recursive in the other sense of Davis [1, p. 171]; weakly equivalent if each is partial recursive in the other in the sense of our definition. If two functions are strongly equivalent they are weakly equivalent; but not conversely by footnote 4 above. The equivalence classes relative to strong (weak) equivalence are called strong (weak) degrees. Not every strong degree contains a full function (footnote 4). Our main result in this note is that the same is true of weak degrees—in fact (cf. footnote 4) that "almost no" (weak) degree contains a full function. This will be established if we can prove the following theorem.

For each set $\alpha$, let $c_\alpha^\theta$ (the semicharacteristic function of $\alpha$) be that function which is 0 on $\alpha$ and undefined elsewhere. The class of all sets $\alpha$ for which some full nonrecursive function is partial recursive in $c_\alpha^\theta$ in the sense of our definition is of first category.\(^6\) A fortiori the same is true of the class of all $\alpha$ for which some full nonrecursive function is partial recursive in $c_\alpha^\theta$ in Davis' sense, and of the class of all $\alpha$ for which $c_\alpha^\theta$ is strongly (weakly) equivalent to some full function.

For the proof, call $f$ partial recursive in $g$ with Gödel-number $i$ if (1) holds where $R$ is the $i$th recursively enumerable relation in some canonical enumeration. If this is so we write $f = \langle \Phi, g \rangle$.\(^6\)

\(^6\) We use the topology standard in recursion theory, i.e. we identify sets (or their characteristic functions) with points of $\{0, 1\}^\omega$, where $\{0, 1\}$ is given the discrete topology. The collection of all classes $\{\alpha \mid \beta \leq \alpha \text{ and } \alpha \cap \gamma = \emptyset\}$, where $\beta$ and $\gamma$ are disjoint finite sets, forms a convenient basis of open classes.
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does not exist for all \(i, g\); in fact it exists if and only if

\[
\gamma_1^{[1]}, \gamma_2^{[1]} \subseteq g, R_i(x, y_1, t_1), R_i(x, y_2, t_2) \rightarrow t_1 = t_2
\]

where \(R_i\) is the \(i\)th recursively enumerable relation.

The theorem will follow if we can show that the class of all \(\alpha\) for which \([\Phi_\alpha^0]\) is full but not recursive is nowhere dense. Let then \(N\) be any basic open class; we seek a subneighborhood \(N_0\) of \(N\) such that

\[
\alpha \in N_0, [\Phi_\alpha^0] \text{ defined and full } \rightarrow [\Phi_\alpha^0] \text{ recursive.}
\]

Let \(N = \{ \alpha | \beta \subseteq \alpha \text{ and } \gamma \cap \alpha = \emptyset \} \). Then \(N_0\) satisfying (2) is defined by cases as follows.

**Case I.** \([\Phi_\alpha^0]\) is full. Then set \(N_0 = N\). For if \(\alpha\) satisfies the hypothesis of (2) we have \(\alpha \subseteq \gamma', c_\alpha \subseteq c_\gamma, [\Phi_\alpha^0] \subseteq [\Phi_\gamma^0]\). But since \([\Phi_\alpha^0]\) is full, \([\Phi_\alpha^0] = [\Phi_\gamma^0]\) and is therefore recursive.

**Case II.** \([\Phi_\alpha^0]\) is defined, but not full. Again set \(N_0 = N\). For as in Case I, \(\alpha \in N_0 \rightarrow [\Phi_\alpha^0] \subseteq [\Phi_\gamma^0]\). But then \([\Phi_\alpha^0]\) is not full either, and (2) is vacuously true.

**Case III.** \([\Phi_\alpha^0]\) undefined. This can only happen if

\[
(\exists y_1 y_2 t_1 t_2)(y_1^{[1]}, y_2^{[1]} \subseteq c_\gamma, R_i(x, y_1, t_1), R_i(x, y_2, t_2), t_1 \neq t_2).
\]

Let \(\delta\) be the union of the domains of \(y_1^{[1]}\) and \(y_2^{[1]}\) (so that \(\delta \subseteq \gamma'\)). Then we can set \(N_0 = \{ \alpha | \beta \cup \delta \subseteq \alpha \text{ and } \gamma \cap \alpha = \emptyset \} \). For \(\alpha \in N_0 \rightarrow \delta \subseteq \alpha \rightarrow y_1^{[1]}, y_2^{[1]} \subseteq c_\alpha \rightarrow [\Phi_\alpha^0]\) undefined; and again (2) holds vacuously.

**Bibliography**