ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF NONHOMOGENEOUS DIFFERENTIAL EQUATIONS

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I. In the differential equation

$$dy/dt = A(t)y + p(t) \tag{1}$$

$A(t)$ is an $n \times n$ matrix with complex-valued elements which are measurable and bounded for $t \geqq 0$, and p is an n-vector with measurable, complex-valued elements. The norm of a vector (matrix) will be denoted by $\|\cdot\|$ and is defined as the sum of the magnitudes of the elements. A vector (matrix) will be called *bounded* if its norm is bounded on $t \geqq 0$ and *convergent* if its elements tend to finite limits as $t \to \infty$. We shall denote by $X(t)$ the (nonsingular) fundamental matrix of solutions of the homogeneous equation

$$dx/dt = A(t)x \tag{2}$$

for which $X(0) = I$, the unit matrix.

THEOREM 1. *Every solution of* (1) *converges for every convergent* $p(t)$ *if and only if* (i) *every solution of* (1) *is bounded for every bounded* $p(t)$, *and* (ii) *the matrix* $Y(t) = \int_0^t X(t)X^{-1}(\tau)d\tau$ *converges. Moreover,*

$$\lim_{t \to \infty} y(t) = \left[\lim_{t \to \infty} Y(t) \right] \left[\lim_{t \to \infty} p(t) \right].$$

This theorem, which has important implications for control systems, appears to have been overlooked. It is closely related to a theorem of Bellman [**1**] and to the Kojima-Schur theorem [**2**]; our proof is patterned on the functional analytic proofs used by Bellman and Cooke. In connection with Theorem 1 we shall prove a generalization of fundamental theorems of Liapunov and Dini-Hukuhara, which in turn imply a classical result of Cesari [**3**, Teorema 8], and give a special result on asymptotic behavior of solutions of systems (1) having periodic coefficient matrices.

II. We prove first the sufficiency of (i) and (ii). As is well known, the solution of (1) satisfying $y(0) = y_0$ is

$$y(t) = X(t)y_0 + \int_0^t X(t)X^{-1}(\tau)e(\tau)d\tau + \int_0^t X(t)X^{-1}(\tau)d\tau \cdot p \tag{3}$$

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where we have put $\lim_{t\to\infty} p(t)=p$ and $e(t)=p(t)-p$.

In [1] it is shown that (i) implies $\int_0^t\|X(t)X^{-1}(\tau)\|d\tau\leqq M$ for all $t\geqq 0$, as well as $\|X(t)\|\to 0$ as $t\to\infty$. For given $\epsilon>0$ there exists $t_1\geqq 0$ such that for all $t>t_1$, $\|e(t)\|<\epsilon/2M$. From this and (3) it follows that

$$\left\|\int_0^t X(t)X^{-1}(\tau)e(\tau)d\tau\right\| < \|X(t)\|\int_0^{t_1}\|X^{-1}(\tau)e(\tau)\|d\tau+\epsilon/2.$$

Furthermore, there exists $t_2\geqq 0$ such that for $t>t_2$, $\|X(t)\| <\epsilon/2\int_0^{t_1}\|X^{-1}(\tau)e(\tau)\|d\tau$. Hence, for $t>\max(t_1, t_2)$,

$$\left\|\int_0^t X(t)X^{-1}(\tau)e(\tau)d\tau\right\| < \epsilon.$$

Thus the first two terms on the right of (3) tend to zero as $t\to\infty$ and (ii) gives the conclusion.

The necessity of (ii) follows by setting $y_0=0$ in (3) and replacing $p(t)$ successively by the columns of the unit matrix. For the necessity of (i) we set $y_0=0$ in (3) and consider the ith element of the solution vector:

$$y_i(t) = \sum_{j=1}^{n}\int_0^t x_{ij}(t;\tau)p_j(\tau)d\tau, \qquad 1\leqq i\leqq n, \tag{4}$$

where x_{ij} denotes the i, jth element of the matrix $X(t)X^{-1}(\tau)$. The function space C of n-tuples $p=(p_1(t), \cdots, p_n(t))$ of complex-valued functions, measurable for $t\geqq 0$ and tending to finite limits as $t\to\infty$, is a Banach space under the norm

$$|p| = \max_{1\leqq i\leqq n}\left\{\sup_{t\geqq 0}|p_i(t)|\right\}.$$

For fixed $t_0\geqq 0$, the transformation $T^i_{t_0}$, defined by

$$T^i_{t_0}p = y_i(t_0), \qquad i=1,\cdots,n,$$

maps C into the Banach space of complex numbers normed by magnitude. Simple estimates show that $T^i_{t_0}$ is bounded, hence continuous, on C. Using the vector $\xi_\lambda=(p_1^\lambda(t), \cdots, p_n^\lambda(t))$, where

$$p_j^\lambda(t) = \begin{Bmatrix}\operatorname{sgn} x_{ij}(t_0;t), & 0\leqq t\leqq\lambda\leqq t_0\\ 0, & \lambda<t,\ t_0<\lambda\end{Bmatrix}, \qquad j=1,\cdots,n,$$

we find that the bound of $T^i_{t_0}$ is $\sum_{j=1}^n\int_0^{t_0}|x_{ij}(t_0;\ \tau)|d\tau$. Indeed, $|\xi_\lambda|=1$, so that by (4)

$$\frac{|T^i_{t_0}\xi_\lambda|}{|\xi_\lambda|} = \sum_{j=1}^{n} \int_0^\lambda |x_{ij}(t_0;\tau)|\,d\tau$$

from which the assertion follows by a simple argument from the definition of supremum. The hypothesis of Theorem 1 implies that $|T^i_t p|$ is uniformly bounded on $t \geqq 0$ for each $p \in C$. All the hypotheses of the Banach-Steinhaus theorem [4, p. 135] are fulfilled at this point so we may conclude that there exist $K_i \geqq 0$ such that

$$K_i = \sup_{p\in C;\, t\geqq 0} \{|T^i_t p|/|p|\}, \qquad i = 1, \cdots, n.$$

From this it follows that

$$\sup_{t\geqq 0} \left\{ \sum_{j=1}^{n} \int_0^t |x_{ij}(t;\tau)|\,d\tau \right\} < \infty, \qquad i = 1, \cdots, n,$$

hence, that

$$\sup_{t\geqq 0} \left\{ \int_0^t \|X(t)X^{-1}(\tau)\|\,d\tau \right\} < \infty. \tag{5}$$

But Bellman [1] has shown that (5) implies (i) and the proof is complete.

III. The following theorem extends a fundamental theorem of Liapunov [5, p. 34] and one of Dini-Hukuhara [5, p. 37] to the property of interest here.

THEOREM 2. *If every solution of* (1) *is convergent for every convergent* $p(t)$, *if* $D(t)$ *is a matrix whose elements are measurable for* $t \geqq 0$ *and if either* (a) $\lim_{t\to\infty} \|D(t)\| = 0$, *or* (b) $\int_0^\infty \|D(t)\|dt < \infty$ *then every solution of*

$$dz/dt = [A(t) + D(t)]z + p(t) \tag{6}$$

converges for every convergent $p(t)$. *Moreover*, $\lim_{t\to\infty} z(t) = \lim_{t\to\infty} y(t)$.

For the proof, we denote by $W(t)$ the fundamental matrix of the homogeneous system obtained from (6) by letting $p(t) \equiv 0$ and we denote by $Q(t)$ the function $\int_0^t W(t)W^{-1}(\tau)d\tau$. Then by a representation like (3) we have

$$Q(t) = \int_0^t X(t)X^{-1}(\tau)D(\tau)Q(\tau)d\tau + Y(t). \tag{7}$$

This follows from the fact that $Q(t)$ is the solution of the equation

$$dQ/dt = [A(t) + D(t)]Q + I$$

for which $Q(0)=0$. The first hypothesis together with (5) and condition (a) implies—by an argument like that for the sufficiency proof of Theorem 1—that the integral on the right of (7) tends to zero as $t\to\infty$. An entirely similar argument establishes the same fact under condition (b). We conclude that $\lim_{t\to\infty} Q(t)=\lim_{t\to\infty} Y(t)$ so that (ii) is satisfied. It remains to show that (i) is satisfied.

To do this, we need the following condition which is known [6] to be equivalent to (i): *there exist positive constants* α, M *such that*

$$\|X(t)X^{-1}(t_0)\| \leqq M \exp[-\alpha(t - t_0)] \tag{8}$$

for all $t\geqq t_0\geqq 0$. We have then

$$W(t)W^{-1}(t_0) = X(t)X^{-1}(t_0) + \int_{t_0}^{t} X(t)X^{-1}(\tau)D(\tau)W(\tau)W^{-1}(t_0)d\tau \tag{9}$$

as may be verified directly. From (8) and (9) we obtain the inequality

$$e^{\alpha(t-t_0)}\|W(t)W^{-1}(t_0)\| \leqq M + \int_{t_0}^{t} Me^{\alpha(\tau-t_0)}\|W(\tau)W^{-1}(t_0)\|\,\|D(\tau)\|d\tau;$$

by a lemma of Bellman [5, p. 35] this implies

$$e^{\alpha(t-t_0)}\|W(t)W^{-1}(t_0)\| \leqq M \exp M\int_{t_0}^{t}\|D(\tau)\|d\tau.$$

That (8) is satisfied follows immediately from this under condition (b); under condition (a) we need only assume t_0 so large that $\sup_{t\geqq t_0}\|D(t)\|=k<\alpha/M$, which shows that (8) is satisfied for $t\geqq\tau\geqq t_0$. A continuity argument then shows that (8) is also satisfied for $t\geqq\tau$ with $t_0\geqq\tau\geqq 0$. This completes the proof of Theorem 2.

The aforementioned result of Cesari follows as an immediate

COROLLARY. *If, in* (6), $A(t)=A$, *a constant matrix all of whose characteristic roots have negative real parts, then for every convergent* $p(t)$ *every solution of* (6) *converges to the vector* $-A^{-1}\lim_{t\to\infty} p(t)$; *moreover,* $\lim_{t\to\infty}(dz/dt)=0$ *when condition* (a) *is satisfied.*

Indeed, in this case $X(t)X^{-1}(\tau)=X(t-\tau)$ and we have

$$Y(t) = \int_0^t X(\tau)d\tau = A^{-1}\int_0^t (dX/d\tau)d\tau$$

from which it follows that

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} Y(t) = - A^{-1}$$

since, as is well known, our hypothesis on A implies $\lim_{t\to\infty} X(t)=0$. The final statement follows by taking appropriate limits in (6).

IV. A primary implication of Cesari's theorem is that when $A(t)$ is a constant matrix or tends to such a matrix for large t, then (i) alone is sufficient to ensure convergent solutions under convergent perturbations. That this is not the case for general $A(t)$ is illustrated by the following theorem and corollary.

THEOREM 3. *If $A(t)$ is periodic of period τ and if condition* (i) *of Theorem* 1 *is satisfied by* (1) *then there exists a periodic matrix $B(t)$, of the same period τ, such that either $Y(t)\equiv B(t)$ or* $\lim_{t\to\infty} \|Y(t)-B(t)\| =0$.

COROLLARY. *Under the hypotheses of Theorem* 3, *for every convergent $p(t)$ every solution of* (1) *has the asymptotic form*

$$y(t) = B(t) \lim_{t\to\infty} p(t) + \epsilon(t) \tag{10}$$

where $\|\epsilon(t)\| = o(1)$.

For the proof, we set $\tau=1$ without loss of generality. The function

$$C(t) = \left[\sum_{i=0}^{\infty} X(t+i)\right] \int_{-1}^{0} X^{-1}(\tau)d\tau \tag{11}$$

is well-defined since the series is uniformly convergent for $t\geqq 0$ as is implied by the hypothesis together with (8). By the same token, the derived series, $\sum_{i=0}^{\infty} (dX(t+i)/dt)$, converges uniformly for $t\geqq 0$ and it follows that $C(t)$ satisfies the homogeneous differential equation

$$dC/dt = A(t)C. \tag{12}$$

It is readily verified that $C(t)$ is also a particular solution of the nonhomogeneous linear difference equation

$$C(t+1) = C(t) - \int_{-1}^{0} X(t)X^{-1}(\tau)d\tau. \tag{13}$$

The matrix $B(t)$ is defined by

$$B(t) = C(t) + \int_{0}^{t} X(t)X^{-1}(\tau)d\tau \tag{14}$$

so that $B(t)$ is a solution of the nonhomogeneous differential equation

$$dB/dt = A(t)B + I \tag{15}$$

and is therefore not a constant matrix.

From (13) and (14) we obtain

$$\begin{aligned} B(t+1) - B(t) = &-\int_{-1}^{0} X(t)X^{-1}(\tau)d\tau + \int_{0}^{t+1} X(t+1)X^{-1}(\tau)d\tau \\ &- \int_{0}^{t} X(t)X^{-1}(\tau)d\tau. \end{aligned} \tag{16}$$

By the change of variable of integration, $\tau=\lambda+1$, we find that the second term of the right hand member of (16) may be written as

$$\int_{0}^{t+1} X(t+1)X^{-1}(\tau)d\tau = \int_{-1}^{t} X(t+1)X^{-1}(\lambda+1)d\lambda. \tag{17}$$

The hypothesis of periodicity implies that [5, p. 58]

$$X(t) = P(t)Q(t)$$

where $P(t)$ is a periodic matrix with the same period as $A(t)$ and $Q(t)$ is the fundamental matrix of an homogeneous system (2) with a constant matrix of coefficients, so that $Q(t)$ satisfies $Q(t+\tau)=Q(t)Q(\tau)$. With this, it follows from (17) that

$$\int_{0}^{t+1} X(t+1)X^{-1}(\tau)d\tau = \int_{-1}^{t} X(t)X^{-1}(\lambda)d\lambda$$

and we have then that (16) becomes

$$B(t+1) - B(t) = -\int_{-1}^{0} X(t)X^{-1}(\tau)d\tau + \int_{-1}^{0} X(t)X^{-1}(\tau)d\tau \equiv 0 \tag{18}$$

so that $B(t)$ is periodic of period 1. We note that (18) holds whether or not

$$\int_{-1}^{0} X^{-1}(\tau)d\tau = 0. \tag{19}$$

Certainly if (19) obtains, $C(0)=0$ and by a uniqueness theorem for equations like (12) it follows that $C(t)$ is the trivial solution of (12) so that by (14), $Y(t)\equiv B(t)$. It is well to note that (13) implies that $C(0)=0$ only if (19) obtains. If $C(0)\neq 0$, then by virtue of (8) and the relation

$$C(t) = X(t)C(0),$$

it follows from (14) that

$$\lim_{t\to\infty} \|Y(t) - B(t)\| = \lim_{t\to\infty} \|C(t)\| = 0$$

and the proof of the theorem is complete. The corollary is an immediate consequence of the theorem and (3). It is noteworthy that, on the basis of the above argument, $B(t)$ may be characterized as that solution of (15) for which

$$B(0) = \left[\sum_{i=0}^{\infty} X(i)\right] \int_{-1}^{0} X^{-1}(\tau) d\tau,$$

where $B(0) = 0$ if and only if (19) holds.

Finally, in precisely the same manner in which Theorem 2 was established, the following theorem can be proved.

THEOREM 4. *If the hypotheses of Theorem* 3 *are satisfied and if* $D(t)$ *is a matrix which satisfies either of the conditions of Theorem* 2 *then* $\lim_{t\to\infty} \|Q(t) - Y(t)\| = 0$; *hence, the solutions of* (1) *and* (6) *are asymptotically equivalent.*

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