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DUKE UNIVERSITY AND
CASE INSTITUTE OF TECHNOLOGY

A GENERALIZATION OF HIRZEBRUCH POLYNOMIAL AND COBORDISM DECOMPOSITION

YASURÔ TOMONAGA

Introduction. In this paper we shall generalize the Hirzebruch polynomial and utilize it for the determination of cobordism components of a compact orientable and differentiable $4k$-manifold.

1. Let X^{4k} be a compact orientable and differentiable $4k$-manifold. According to Thom's theorem [1] such a manifold is "cobordant" with a polynomial of the complex projective spaces except torsion, i.e.

$$(1.1) \quad X^{4k} \approx \sum_{i_1+\cdots+i_t=k} A^k_{i_1+\cdots+i_t} P_{2i_1}(c) \cdots P_{2i_t}(c) \text{ mod torsion}$$

where $P_i(c)$ denotes the complex projective space of the complex dimension i and A's denotes some rational numbers. The Hirzebruch polynomial is defined as follows [2]

$$(1.2) \quad \prod_i \frac{(\gamma_i)^{1/2}}{\operatorname{tgh}(\gamma_i)^{1/2}} = \sum_{i=0}^{\infty} L_i(p_1, \cdots, p_i), \qquad \sum_{i=0}^{\infty} p_i = \prod_i (1+\gamma_i)$$

where p_i denotes the Pontryagin class of dimension $4i$. It is well known that

$$(1.3) \qquad L_i(p_1, \cdots, p_i)[P_{2i}(c)] = 1,$$

from which we have

$$(1.4) \quad L_k(p_1, \cdots, p_k)[X^{4k}] = \text{index of } X^{4k} = \sum_{i_1+\cdots+i_t=k} A^k_{i_1\cdots i_t}.$$

We generalize (1.2) as follows:

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$$
\begin{aligned}
&\prod_i \frac{(\gamma_i)^{1/2}}{\operatorname{tgh}(\gamma_i)^{1/2}}(1 + y \operatorname{tgh}^2(\gamma_i)^{1/2}) = \sum_i \Gamma_i(y, p_1, \cdots, p_i) \\
&= 1 + \left(y + \frac{1}{3}\right)p_1 + \left\{p_2 y^2 + \frac{1}{3}(4p_2 - p_1^2)y + \frac{1}{45}(7p_2 - p_1^2)\right\} \\
&\quad + \left\{p_3 y^3 + \frac{1}{3}(6p_3 - p_1p_2)y^2 + \frac{1}{15}(17p_3 - 8p_1p_2 + 2p_1^3)y \right. \\
&\quad \left. + \frac{1}{3^3\cdot 5\cdot 7}(62p_3 - 13p_1p_2 + 2p_1^3)\right\} \\
&\quad + \cdots
\end{aligned} \tag{1.5}
$$

and

$$
\begin{aligned}
&\prod_i \frac{(\gamma_i)^{1/2}}{\operatorname{tgh}(\gamma_i)^{1/2}}(1 + y \operatorname{tgh}^2(\gamma_i)^{1/2})^{-1} = \sum_i \Lambda_i(y, p_1, \cdots, p_i) \\
&= 1 + \left(\frac{1}{3} - y\right)p_1 \\
&\quad + \left\{(p_1^2 - p_2)y^2 + \frac{1}{3}(p_1^2 - 4p_2)y + \frac{1}{45}(7p_2 - p_1^2)\right\} \\
&\quad + \left\{-(p_3 - 2p_1p_2 + p_1^3)y^3 + (-2p_3 + 3p_1p_2 - p_1^3)y^2 \right. \\
&\quad + \frac{1}{15}(-17p_3 + 8p_1p_2 - 2p_1^3)y \\
&\quad \left. + \frac{1}{3^3\cdot 5\cdot 7}(62p_3 - 13p_1p_2 + 2p_1^3)\right\} \\
&\quad + \cdots .
\end{aligned} \tag{1.6}
$$

It follows from (1.4) and (1.5) that

$$
\begin{aligned}
&\text{(i)} \quad \Gamma_i(0, p_1, \cdots, p_i)[X^{4i}] = \text{index of } X^{4i}, \\
&\text{(ii)} \quad \Gamma_i(1, p_1, \cdots, p_i)[X^{4i}] = 2^{2i}(\text{index of } X^{4i}), \\
&\text{(iii)} \quad \Gamma_i(-1, p_1, \cdots, p_i)[X^{4i}] = A\text{-genus of } X^{4i} \text{ [2, p. 14]}.
\end{aligned} \tag{1.7}
$$

Moreover we can prove that $\Gamma_i(y, p_1, \cdots, p_i)[X^{4i}]$ has integral coefficients. The complete proof will be given in another paper but the proof is easy in the case of an almost complex split manifold because $\Gamma_i(X^{4i})$ decomposes into many virtual indices [**2**, p. 87].

2. Next we consider the application of our multiplicative series for the determination of cobordism coefficients. We have from (1.1)

$$\Lambda_k(y, p_1, \cdots, p_k)\,[X^{4k}]$$

$$(2.1)\qquad = \sum_{i_1+\cdots+i_t=k} A^{k}_{i_1\cdots i_t}\Lambda_{i_1}[P_{2i_1}(c)]\cdots\Lambda_{i_t}[P_{2i_t}(c)] \qquad (k \leqq 4).$$[1]

Comparing the coefficients of y^a's $(a=0, \cdots, k)$ we have

$$(2.2)\qquad A^{2}_{2} = \frac{1}{5}(-2p_2 + p_1^2)[X^8], \qquad A^{2}_{11} = \frac{1}{9}(5p_2 - 2p_1^2)[X^8],$$

$$A^{3}_{3} = \frac{1}{7}(3p_3 - 3p_1p_2 + p_1^3)[X^{12}],$$

$$(2.3)\qquad A^{3}_{21} = \frac{1}{15}(-21p_3 + 19p_1p_2 - 6p_1^3)[X^{12}],$$

$$A^{3}_{111} = \frac{1}{27}(28p_3 - 23p_1p_2 + 7p_1^3)[X^{12}],$$

$$A^{4}_{4} = \frac{1}{9}(-4p_4 + 4p_1p_3 + 2p_2^2 - 4p_1^2p_2 + p_1^4)[X^{16}],$$

$$A^{4}_{31} = \frac{1}{21}(36p_4 - 33p_1p_3 - 18p_2^2 + 33p_1^2p_2 - 8p_1^4)[X^{16}],$$

$$(2.4)\qquad A^{4}_{22} = \frac{1}{25}(18p_4 - 18p_1p_3 - 7p_2^2 + 16p_1^2p_2 - 4p_1^4)[X^{16}],$$

$$A^{4}_{211} = \frac{1}{45}(-180p_4 + 159p_1p_3 + 80p_2^2 - 150p_1^2p_2 + 36p_1^4)[X^{16}],$$

$$A^{4}_{1111} = \frac{1}{81}(165p_4 - 137p_1p_3 - 70p_2^2 + 127p_1^2p_2 - 30p_1^4)[X^{16}].$$

3. Next we consider the case where a X^{4k} is a submanifold of X^{4k+2r} where we assume that both manifolds be compact orientable and differentiable. Let X^{4k} be determined by the cohomology classes $v_1, \cdots, v_r \in H^2(X^{4k+2r}, Z)$. Then we can determine the cobordism coefficients of X^{4k} by v's and the Pontryagin classes of X^{4k+2r} as follows:

$X^8 \subset X^{10}$,

$$A^{2}_{2} = \frac{1}{5}(-v^5 - 2vp_2 + vp_1^2)[X^{10}],$$

(3.1)

$$A^{2}_{11} = \frac{1}{9}(3v^5 - v^3p_1 + 5vp_2 - 2vp_1^2)[X^{10}],$$

---

[1] Γ_k is also available for this purpose but for $k \leqq 3$.

$X^{12} \subset X^{14}$,

$$
\begin{aligned}
A^3_3 &= \frac{1}{7}\{-v^7 + (3p_3 - 3p_1p_2 + p_1^3)v\}[X^{14}], \\
A^3_{21} &= \frac{1}{15}\{8v^7 - v^5p_1 + v^3(2p_2 - p_1^2) \\
&\qquad + v(-21p_3 + 19p_1p_2 - 6p_1^3)\}[X^{14}], \\
A^3_{111} &= \frac{1}{27}\{-12v^7 + 3v^5p_1 + (2p_1^2 - 5p_2)v^3 \\
&\qquad + (28p_3 - 23p_1p_2 + 7p_1^3)v\}[X^{14}],
\end{aligned}
\tag{3.2}
$$

$X^8 \subset X^{12}$,

$$
\begin{aligned}
A^2_2 &= \left\{-\frac{1}{5}(v_1^5v_2 + v_2^5v_1) + \frac{1}{5}(p_1^2 - 2p_2)v_1v_2\right\}[X^{12}], \\
A^2_{11} &= \left\{\frac{1}{3}(v_1^5v_2 + v_2^5v_1) + \frac{1}{9}v_1^3v_2^3 - \frac{1}{9}(v_1v_2^3 + v_2v_1^3)p_1 \right. \\
&\qquad \left. + \frac{1}{9}(5p_2 - 2p_1^2)v_1v_2\right\}[X^{12}].
\end{aligned}
\tag{3.3}
$$

The method used here was as follows [2, p. 87]: From (2.1) replaced by Γ_k and the relation

$$
\begin{aligned}
&\Gamma_k(y, p_1, \cdots, p_k)[X^{4k}] \\
&= \left[\kappa^{4k+2r}\left\{\left(\frac{\operatorname{tgh} v_1}{1 + y \operatorname{tgh}^2 v_1}\right) \cdots \left(\frac{\operatorname{tgh} v_r}{1 + y \operatorname{tgh}^2 v_r}\right)\right.\right. \\
&\qquad \left.\left. \cdot \sum_i \Gamma_i(y, p_1, \cdots, p_i)\right\}\right][X^{4k+2r}]
\end{aligned}
\tag{3.4}
$$

we obtain (3.1)-(3.3) by comparing the coefficients of y^a's.

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Utsunomiya University, Japan