1. **Transformation.** Let $\mathcal{R}$ be a $\sigma$-algebra\(^2\) of subsets of $X$, and $\emptyset$ the set of all probability measures $P$ on $\mathcal{R}$. Let $T$ transform $\emptyset$ into itself. For certain sets $E \subseteq \mathcal{R}$, knowledge of $P$ throughout $E$ (i.e., for all subsets of $E$ belonging to $\mathcal{R}$) determines $TP$ throughout $E$. The class of sets having this property will be denoted by $E_T$, or better, since $T$ will be fixed, by $E$. Evidently $E$ contains $0$, $X$, and the complements of atoms. We show that if $E$ is sufficiently large, then $T$ is a linear combination of the identity and a constant. There are applications to the theory of learning and to political theory \([1;3;4;6]\).

**Theorem 1.** (A) If $E$ contains an algebra $A$ whose Borel extension is $\mathcal{R}$, and if $|\mathcal{R}| > 4$, then $TP = \alpha P + (1-\alpha)P_0$, where $\alpha \leq 1$ and $P_0 \in \emptyset$.

(B) The converse is true with no restriction on $\mathcal{R}$.

(C) If $\mathcal{R}$ is infinite, then $\alpha \geq 0$.

In the political interpretation, the elements of $X$ are parties (or political positions). $P$ is the distribution of voters, $T$ is the electoral mechanism, and $TP$ the distribution of seats in the legislature. If $T$ is the identity, the mechanism is Proportional Representation. If $T$ is a constant, the political complexion of the legislature is fixed by law. It will be seen from Theorem 3 that $E \subseteq E$ means that if the complement $-E$ unites in a coalition, the effect is independent of whether this occurs before or after the election. $|\mathcal{R}| > 4$ means essentially that there are more than two parties. Part (A) of the theorem is not true for $|\mathcal{R}| = 4$.

In learning theory, $P$ is a probability distribution of responses, and $TP$ is a new distribution resulting from a learning experience. If $T$ is the identity, there is no learning. If $T$ is a constant, this is one-trial learning.

Bush, Mosteller, and Thompson \([4]\) proved an equivalent theorem for the case $\mathcal{R}$ finite and $E = \mathcal{R}$ (Corollary 3 of Theorem 3). Some of their ideas are used in the proof.

Denote by $B$ the class of sets $E$ such that $P(E) = Q(E)$ implies $TP(E) = TQ(E)$, for all $P$, $Q \in \emptyset$. The importance of $B$ is that for

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\(^2\) Borel field. We follow the terminology of \([5]\). In addition, the Borel extension of a class $M$ of sets is the smallest $\sigma$-algebra containing $M$. This is the same as $S(M)$ if $X$ is the countable union of sets in $M$. 

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each proper set $E \in B$, there is a function $\gamma_E$ mapping $[0, 1]$ into itself such that $TP(E) = \gamma_E[P(E)]$. We have also $\gamma_0(0) = 0$.

For any class $S$ of subsets of $X$, let $S^*$ denote the class of sets $E$ for which the complement $-E \in S$.

**Proposition 1.** $E \cap E^* \subseteq B$.

**Proof.** Let $E, -E \in E$. Let $P(E) = Q(E)$ for two members $P, Q$ of $\emptyset$. Define $P' \in \emptyset$ as follows. For $A \in R$, let $P'(A) = P(A - E) + A(c)P(E)$, where $c \in E$ and $A(x)$ is the characteristic function of $A$. Define $Q'$ similarly. We have $P' \equiv Q'$ on $E$. Also $P \equiv P'$ and $Q \equiv Q'$ on $-E$. Hence $TP' = TQ'$ on $E$, while $TP \equiv TP'$ and $TQ \equiv TQ'$ on $-E$. In particular, the last two equations are true for $-E$ itself, and, taking complements, also for $E$. We have $TP(E) = TP'(E) = TQ'(E) = TQ(E)$, proving $E \in B$.

Since $A$ is an algebra, $A = A^*$. Thus $B \supseteq E \cap E^* \supseteq A \cap A^* = A$, and so $B \supseteq A$.

Define the set function $u$ on the class $A - \{X\}$ as follows: $u(E) = \gamma_E(0)$. Using the fact that $TP$ is a measure for each $P$, and choosing $P$ so as to vanish on the appropriate sets, it is easy to show that $u$ is a measure on the semiring $A - \{X\}$, and therefore extends uniquely to a measure $u$ on $R [5; 7]$. Evidently $u \leq 1$ on $A - \{X\}$, but it would be incorrect to infer that $u(X) \leq 1$.

**Proposition 2.** Let $E \cap F = 0$; $E, F, E \cup F$ proper sets in $B$; $x, y, x + y \in [0, 1]$. Then $\gamma_{E \cup F}(x + y) = \gamma_E(x) + \gamma_F(y)$.

**Proof.** Using all the hypotheses, it is easy to show that there is a probability measure $P$ with $P(E) = x$ and $P(F) = y$. For this $P$,

$\gamma_{E \cup F}(x + y) = TP(E \cup F) = TP(E) + TP(F) = \gamma_E(x) + \gamma_F(y)$.

If $E$ and $F$ are proper sets in $A$, let $E \sim F$ denote the statement that $\gamma_E(x) - u(E) = \gamma_F(x) - u(F)$ for all $x \in [0, 1]$. $\sim$ is an equivalence relation.

**Proposition 3.** The relation $\sim$ is universal on the proper sets in $A$.

**Proof.** (1) Let $E \subseteq B$, where $\subseteq$ denotes proper inclusion. By Proposition 2, with $F = B - E$ and $y = 0$, $\gamma_B(x) = \gamma_E(x) + \gamma_{B - E}(0)$. Letting $x = 0$, $\gamma_B(0) = \gamma_E(0) + \gamma_{B - E}(0)$. Subtracting, we have $E \sim B$.

(2) If $E \cap F = 0$ and $E \cup F \neq X$, then $E \sim E \cup F \sim F$ by (1).

(3) If $E, F$ are incomparable and $E \cap F \neq 0$, then $E \sim E \cap F \sim F$ by (1).

(4) This leaves only the case $F = -E$. For the first time, we invoke the hypothesis $|R| > 4$, which easily implies $|A| > 4$. Hence $E$ or $F$
must have a proper subset, say \( E \supseteq A \). Then \( E \sim A \) by (1) and \( A \sim F \) by (2).

In view of Proposition 3 and \( |A| > 2 \), the equation

\[
\gamma(x) = \gamma_E(x) - u(E) \quad \text{for } E \subseteq A - \{0, X\}
\]
defines \( \gamma(x) \) uniquely. \( \gamma \) maps \([0, 1]\) into \([-1, 1]\).

**Proposition 4.** \( \gamma(x) = \alpha x \), with \( \alpha \leq 1 \).

**Proof.** Let \( x, y, x+y \in [0, 1] \). Choose \( E, F \) so that \( E, F, E \cup F \) are proper sets in \( A \), and so that \( E \cap F = \emptyset \). Here we have used \(|R| > 4\) for the second and last time. By Proposition 2 and the definitions of \( u \) and \( \gamma \),

\[
\gamma(x + y) + u(E \cup F) = \gamma_{E \cup F}(x + y) = \gamma_E(x) + \gamma_F(y)
\]

\[
= \gamma(x) + u(E) + \gamma(y) + u(F).
\]

Since \( u \) is additive, we conclude that \( \gamma(x+y) = \gamma(x) + \gamma(y) \). A bounded function of this type is of the stated form. The proof in [2] can be adapted. Obviously, \( \alpha \leq 1 \).

Thus \( TP(E) = \alpha P(E) + u(E) \) for all \( E \) in the semiring \( A - \{X\} \). If \( \alpha \geq 0 \), then \( \alpha P + u \) is a measure on \( R \), equal to \( TP \) on \( A - \{X\} \), and therefore on \( R \). If \( \alpha \leq 0 \), then \( TP - \alpha P \) is a measure on \( R \), equal to \( u \) on \( A - \{X\} \), and therefore on \( R \). In either case \( TP = \alpha P + u \) on \( R \).

In passing, note that

\[
(1) \quad 1 = TP(X) = \alpha + u(X).
\]

If \( \alpha = 1 \), then \( TP \equiv P \). If \( \alpha < 1 \), define \( P_0 \) by \((1-\alpha)P_0 = u \). The main assertion (A) of Theorem 1 follows.

Assertion (B) is immediate, taking \( A = R \). For (C) we require a simple result from set theory. We omit the proof, which is not difficult.

**Lemma.** If \( A \) is an infinite algebra of subsets of \( X \), then \( X \) is the union of a monotone sequence of sets of \( A - \{X\} \).

To resume the proof of (C), the infinite cardinality of \( R \) implies the same for \( A \). Then we have \( u(X) = \lim_{n \to \infty} u(E_n) \) for sets \( E_n \subseteq A - \{X\} \). But \( u \leq 1 \) on \( A - \{X\} \), and therefore \( u(X) \leq 1 \). With (1), we have \( \alpha \geq 0 \).

For applications to special cases, we need the following closure properties of \( E \), which are of independent interest.

**Theorem 2.** (A) If \( E, F \subseteq E \), and \( E \cup F \neq X \), then \( E \cap F \subseteq E \).

(B) \( E \) is closed with respect to countable union.
Proof. (A) Let \( P = Q \) on \( E \cap F \). Without loss of generality, assume \( P(E) \leq Q(E) \). Define a new probability measure \( P' \) as equal to \( P \) on \( E \) (i.e., throughout \( E \)), equal to \( Q \) on \( F - E \), and arbitrary on \( X - E - F \) except that \( P'(E) + P'(F - E) + P'(X - E - F) = 1 \). For other sets, \( P' \) is defined by additivity. In verification that the values assigned on \( X - E - F \) are feasible, we observe that this set is not empty and that \( P'(E \cup F) \leq Q(E \cup F) \leq 1 \).

Now \( P = P' \) on \( E \) and \( Q = P' \) on \( F \). Hence \( TP = TP' \) on \( E \) and \( TQ = TP' \) on \( F \). The last two identities are true, therefore, on \( E \cap F \). Hence \( TP = TQ \) on \( E \cap F \), and \( E \cap F \subseteq E \).

We remark that when \( E \cap F = X \), (A) is false in the strong sense that given such overlapping incomparable \( E \), \( F \), there exists a \( T \) for which \( E \) and \( F \) are in \( E_T \), but \( E \cap F \) is not.

(B) Let \( P = Q \) on \( E = \bigcup_i E_n \), where \( E_n \subseteq E \). Then \( P = Q \) on \( E_n \), which implies \( TP = TQ \) on \( E_n \), for each \( n \). Let \( \{ F_n \} \) be a disjoint sequence having the same partial unions as \( \{ E_n \} \). We have \( TP = TQ \) on \( F_n \), since \( F_n \subseteq E_n \). Then \( TP = TQ \) on \( E \) by countable additivity, and \( E \subseteq E \).

The hypothesis of Theorem 1 may be expressed in two parts:

(I) \(|R| > 4\), \( E \) contains a class \( S \) whose Borel extension is \( R \), and \( X \subseteq S \).

(II) \( S \) is a ring.

Proposition 5. In Theorem 1, (II) can be weakened to: \( S \) is a semi-ring.

Proof. The class of finite disjoint unions of elements of \( S \) is a ring \([5]\). Since it contains \( S \), this ring generates \( R \) and contains \( X \). By Theorem 2B, \( E \) contains the ring.

Examples. In all of these, let \( R \) be the class of Borel sets.

(i) \( X \) = the real line. Let \( E \) contain all intervals \([a, \beta)\). (Here and in the following it would suffice to take \( \alpha \) and \( \beta \) rational.) Then \( E \) contains also \([a, \infty)\) and \((-\infty, a]\). With \( 0 \) and \( X \), these finite and semi-infinite intervals constitute a semiring. Proposition 5 applies, and \( TP = \alpha P + (1-\alpha)P_0 \) with \( 0 \leq \alpha \leq 1 \) as in Theorem 1. This is equally true if \( E \) is assumed instead to contain all proper closed intervals.

(ii) (a) \( X = (0, 1) \), (b) \( X = [0, 1] \), (c) \( X = [0, 1) \). Similar to Example (i).

(iii) \( X \) = Euclidean \( n \)-space \((n > 1)\). Let \( E \) contain all half spaces \( \{x: x_i \geq \alpha\} \) and \( \{x: x_i < \alpha\} \). Then \( E \) contains all finite intersections of these sets. (This implication is false for \( n = 1 \).) With \( X \) added, these constitute a semiring, Proposition 5 applies, and \( T \) has the form
stated in Theorem 1. This is true also if $E$ is assumed to contain all slices \( \{x: x_i \in [\alpha, \beta]\} \), or alternatively all cells
\[ \{x: x_i \in [\alpha_i, \beta_i], \ i = 1, \ldots, n\}. \]

2. **Combination.** Bush and Mosteller [3] raised the question in learning theory of whether a set $E$ could be shrunk to a point without making $T$ ambiguous on the reduced space. More precisely, let $E \subseteq \mathbb{R}$. A transformation $C$ of $\emptyset$ into itself is called a combination of $E$ if it satisfies
\[
(C1) \quad CP = P \text{ on } -E \\
(C2) \quad P(E) = Q(E) \implies CP = CQ \text{ on } E.
\]

For example, let $c \in E$, and let
\[
(2) \quad CP(A) = P(E - A) + A(c)P(E) \quad \text{ for each } A \subseteq \mathbb{R}.
\]

We say that $E \subseteq C$ ($E$ is combinable) if for each combination $C$ of $E$, and for each $P \in \mathcal{P}$, we have
\[
(3) \quad CTP = CTQ.
\]

In learning theory, (3) is called the *Combining of Classes* condition.

**Theorem 3.** $C = E^*.$

**Proof.** Let $P \in C$, and $C$ be a combination of $E$. We observe first that (C1) and (C2) imply
\[
(C3) \quad CP = CQ \text{ if and only if } P = Q \text{ on } -E.
\]

Now let $P = Q$ on $-E$. Then $CP = CQ$. Hence $CTP = CTCP = CTQ$. Then a second use of (C3) yields $TP = TQ$ on $-E$. This proves $C \subseteq E^*.$

Let $-E \subseteq E$. Let $C$ combine $E$. Then $P = CP$ on $-E$, and therefore $TP = TCP$ on $-E$. By (C3), $CTP = CTCP$. Thus $E^* \subseteq C$.

**Corollary 1.** *In Theorem 1, the hypothesis that $E$ contains the algebra $A$ can be replaced by $C \supseteq A$.***

**Corollary 2.** (A) The union of two overlapping sets of $C$ is in $C$.
(B) $C$ is closed with respect to countable intersection.

**Corollary 3.** *Let $X$ be finite, $|X| > 2$, let $R$ be the class of all subsets of $X$, and $C = R$. Then $TP = \alpha P + (1 - \alpha)P_0$.***

This is the Bush-Mosteller-Thompson theorem [4] mentioned.
earlier. Bush and Mosteller [3] showed that $\alpha(|X| - 1) \geq -1$. This bound is attained.

Regarding Example (ii)(c) as the real numbers modulo 1, let $C$ contain all intervals $[\alpha, \beta)$, naturally including the case $\alpha < 1 < \beta$. With 0, this class is a semiring $S$. Since $S = S^*$, also $E \supseteq S$, so that Proposition 5 applies, and $T$ has the familiar form of Theorem 1. In Example (i) (the real line) the corresponding implication is false, even with the additional assumption that $C$ contains all semi-infinite intervals, and similarly for Example (ii). To prove this, we use

**Proposition 6.** Let $T$ be a combination of $E$ of type (2). Then

$$C_T = \left[ \{ \{ c \} \} + \{ E \} - \{ -E \} \right] \cap R.$$

(For any subset $S$ of $X$, $\{ S \}$ and $\{ S \}^*$ denote respectively the class of subsets of $S$ and the class of supersets of $S$.) The proof is omitted.

Returning to Example (i), let $T$ be that combination of $E = [c, \infty)$ of type (2) which concentrates $P(E)$ at $c$. We see that $C$ contains all the finite and infinite intervals mentioned above, but that intervals $[\alpha, \beta)$ containing $c$ are not in $E$, and the conclusion of Theorem 1 is false here.

3. **Partition.** A related problem, motivated by learning theory and political theory, is the following. For $n > 1$, let $\mathcal{P}_n$ denote the class of all partitions of $X$ into exactly $n$ nonempty parts $X_i$, and let $\mathcal{E}_n$ denote the subclass of partitions (called combinable) for which the $n$-tuple $[P(X_1), \ldots, P(X_n)]$ uniquely determines $[TP(X_1), \ldots, TP(X_n)]$. It is not difficult to show that if each $X_i \in C$, then $(X_1, \ldots, X_n) \in \mathcal{E}_n$. The converse is false, so that the latter statement is actually weaker than the former. Despite this, we have

**Theorem 4.** If $\mathcal{E}_n = \mathcal{P}_n$ for some $n < \log_2 |R|$, then $E = C = R$, and $TP = \alpha P + (1 - \alpha) P_0$.

**Proof.** First we show that $E \in \mathcal{E}$ for all $E$ divisible into $n - 1$ (proper) parts. We can assume $E \neq X$. Let $P = Q$ on $E$, and let $A \subseteq E$. We can express $A$ as $\bigcup_i A_i$, where either $a = n - 1$ or each $A_i$ is atomic. If $a < n - 1$, then $E - A = \bigcup_{i+1} A_i$ by the hypothesis on $E$. In either case, $P(A_i) = Q(A_i)$ for $i = 1, \ldots, n - 1$ and $P(-E) = Q(-E)$. Hence $TP(A_i) = TQ(A_i)$. Summing from 1 to $a$, $TP(A) = TQ(A)$, and $E \in \mathcal{E}$.

Evidently $E \in \mathcal{E}$ is proved unless $E$ consists of the union of fewer than $n - 1$ atoms. Let $E$ be the union of $n - 2$ atoms. Since $|R| > 2^n$, $-E$ has three parts, $A, B, C$. Then $E \cup A$ and $E \cup B$ are in $E$, their
union is not \( X \), and so their intersection \( E \) is in \( E \) by Theorem 2A. Similarly for \( n - 3 \), etc. Thus \( E = R \), and the remaining statements follow from Theorems 1 and 3.

The theorem is false for \( n \geq \log_2 |R| \).

Let \( \mathcal{S}_n \) denote the class of partitions of \( X \) into \( n \) nonempty parts \( X_i \), each of which is in a fixed semiring \( S \) whose Borel extension is \( R \). (The notation is suggested by examples where \( S \) consists of intervals.)

When \( X \) is the real line, and \( S \) the class of intervals \([\alpha, \beta), (-\infty, \alpha), [\alpha, \infty)\), the example at the end of §2 shows that \( 0 \neq \mathcal{S}_n \subseteq \mathcal{C}_n \) for all \( n \) does not imply \( E = R \). The same is true for \( X \) a finite interval. The situation is different for a circle.

**Theorem 5.** Let \( X \) be the set of real numbers modulo 1, and \( S \) the class of intervals \([\alpha, \beta)\). If \( \mathcal{S}_n \subseteq \mathcal{C}_n \) for some \( n \), then \( E = C = R \), and \( TP = \alpha P + (1 - \alpha) P_0 \).

**Proof.** If \( n = 2 \), then evidently \( S \subseteq B \). With the single exception of Proposition 3, the proof of Theorem 1A applies, with the semiring \( S \) replacing the algebra \( A \). Proposition 1 is superfluous. We show now that the conclusion of Proposition 3 holds also in the present context. All intervals mentioned are proper, i.e., not 0 or \( X \).

(1) Let \( I_1 \subseteq I \). If \( I - I_1 \) is an interval, then \( I_1 \sim I \) as in Proposition 3. If \( I - I_1 \) is not an interval, then it is the union of two disjoint intervals \( I_2 \) and \( I_4 \). Moreover, \( I_1 \cup I_2 \) is an interval. Thus \( I_1 \sim I_1 \cup I_2 \sim I \).

(2) If \( I_1 \cap I_2 = 0 \) and \( I_1 \cup I_2 \neq X \), then there is a proper interval \( I \) containing \( I_1 \cup I_2 \). Then \( I_1 \sim I \sim I_2 \) by (1).

Thus (1) and (2) in the proof of Proposition 3 are true in our present case. (3) and (4) apply unchanged. (This proof that \( S \subseteq B \) implies the linearity of \( T \) is valid also for \( X = \) the real line with \( S \) all \([\alpha, \beta)\), and for \( X = \) Euclidean \( n \)-space with \( S \) all semiclosed cells.) This completes the proof for \( n = 2 \).

Next, let \( n > 2 \). Note that \( P \equiv Q \) on \( I \) if and only if \( P \equiv Q \) for all subintervals of \( I \) touching an end point. Hence \( I \subseteq E \) if and only if the equality of \( P \) and \( Q \) for all such subintervals implies the same for \( TP \) and \( TQ \).

Let \( P \equiv Q \) on \( I \), and let \( I_1 \) be a subinterval touching an end point. Write the interval \( I - I_1 \) as the disjoint union \( U_s I_s \), and let \( I_s = -I_1 \). (Here we have used the fact that \( S \equiv S^* \).) We have \( \langle I_1, \cdots, I_n \rangle \subseteq \mathcal{C}_n \), and \( P(I_j) = Q(I_j) \) for all \( j \). Hence \( TP(I_j) = TQ(I_j) \) for all \( j \), and in particular for \( j = 1 \). Since \( I_1 \) was arbitrary, this proves \( I \subseteq E \). Thus \( S \subseteq E \).

Using \( S = S^* \) again, we have \( S \subseteq E \cap E^* \). By Proposition 1, \( S \subseteq B \), and the first part of the proof applies.
References


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