A SYMMETRY THEOREM FOR THE DIFFERENTIAL IDEAL $[uv]$

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1. **Introduction.** Let $F[uv]$ be a Ritt algebra in the indeterminates u and v, and let $[uv]$ be the differential ideal generated by the form $X=uv$. If $P=UV$ is a power product (pp.) in u_i and v_j (the subscripts indicate derivatives) and contains no v_k, $k<d_1$ (d_1 is the degree of U), then $P\not\equiv 0\ [uv]$. Such pp. are called α-terms, and, in particular, a pp. in u alone, a pp. in v alone, and unity are α-terms. All other pp. may be reduced modulo $[uv]$ to a linear combination of α-terms by H. Levi's reduction process [**1**; **2**]. Levi's methods provide an answer to the question of whether or not a pp. is in the ideal $[uv]$ because a linear combination of α-terms is congruent to zero modulo $[uv]$ if and only if all the coefficients are zero. Both the reduction process and the above definitions do not make use of the natural symmetry of the ideal $[uv]$. A pp. $P=UV$ of signature (d_1, d_2) and weight $w=d_1d_2$ is reduced to a multiple of the α-term $u_0^{d_1}v_{d_1}^{d_2}$, but, by interchanging the roles of u and v, one could reduce P to a multiple of the term $u_{d_2}^{d_1}v_0^{d_2}$. In certain of the problems suggested by J. F. Ritt [**3**], it would be convenient to know the relationship between $u_0^{d_1}v_{d_1}^{d_2}$ and $u_{d_2}^{d_1}v_0^{d_2}$ so that both types of reductions could be used. The purpose of this note is to exhibit the exact relationship between $u_0^dv_d^d$ and $u_d^dv_0^d$ so that for a pp. of signature (d, d) and weight $w=d^2$, the u_i and v_i may be interchanged.

2. **Symmetry theorems.** Let $P=UV$ have signature (d_1, d_2) and weight $w=w_1+w_2$. A theorem of H. Levi states that if $w<d_1d_2$, then $P\equiv 0[uv]$. Special cases of this theorem are stated for easy reference as

LEMMA 2.1. (a) *If* $P_k=u_0u_1\cdots u_{k-1}u_k^2v_1v_2\cdots v_{k+1}$, *then* $P_k\equiv 0[uv]$. (b) *If* $P_k=u_0u_1\cdots u_kv_1v_2\cdots v_{k-1}v_k^2$, *then* $P_k\equiv 0[uv]$.

PROOF. (a) The signature of P_k is $(k+2, k+1)$ and the weight is k^2+3k+1, hence $w<d_1d_2$. The proof of (b) is similar.

THEOREM 2.2.

$$u_0u_1\cdots u_jv_1v_2\cdots v_{j+1}\equiv(-1)^{j+1}u_1\cdots u_{j+1}v_0\cdots v_j\ [uv].$$

PROOF. For $j=0$, $[uv]_1=u_0v_1+u_1v_0\equiv 0[uv]$, hence $u_0v_1\equiv -u_1v_0[uv]$. Assume that the theorem is true for all values less than j. Replacing

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u_jv_{j+1} by the other terms in the $(2j+1)$st derivative of $[uv]$, we have

$$u_0u_1\cdots u_jv_1v_2\cdots v_{j+1}\equiv -u_0u_1\cdots u_{j-1}v_1v_2\cdots v_j \times \sum_{k=0;\,k\neq j}^{2j+1}\frac{\binom{2j+1}{k}}{\binom{2j+1}{j}}u_kv_{2j+1-k}\ [uv].$$

Except for the term $k=j+1$, each term of the sum is zero modulo $[uv]$ by Lemma 2.1. The induction hypothesis applies to the term $k=j+1$, and noting that

$$\frac{\binom{2j+1}{j+1}}{\binom{2j+1}{j}}=1,$$

the proof is concluded.

LEMMA 2.3. *If* $j>0$ *and* $0\leqq t\leqq j-1$, *then*

$$u_0u_1\cdots u_{j-t-2}u_{j-t-1}^{r}u_{j-t}^{t-r+2}v_1v_2\cdots v_{j-t-1}v_{j-t+r-1}v_{j+1}^{t+1}$$
$$\equiv -\frac{\binom{2j-2t+r-1}{j-t-1}}{\binom{2j-2t+r-1}{j-t}}$$
$$\times\ u_0u_1\cdots u_{j-t-2}u_{j-t-1}^{r+1}u_{j-t}^{t-r+1}v_1v_2\cdots v_{j-t-1}v_{j-t+r}v_{j+1}^{t+1}\ [uv]$$

for $0<r\leqq t+1$.

PROOF. Replace $u_{j-t}v_{j-t+r-1}$ by the other terms in the $(2j-2t+r-1)$th derivative of $[uv]$ and get the congruence

$$u_0u_1\cdots u_{j-t-2}u_{j-t-1}^{r}u_{j-t}^{t-r+2}v_1v_2\cdots v_{j-t-1}v_{j-t+r-1}v_{j+1}^{t+1}$$
$$\equiv u_0u_1\cdots u_{j-t-2}u_{j-t-1}^{r}u_{j-t}^{t-r+1}v_1v_2\cdots v_{j-t-1}v_{j+1}^{t+1}$$
$$\times\sum_{k=0;\,k\neq j-t}^{2j-2t+r-1}A_{k,r}u_kv_{2j-2t+r-1-k}\ [uv],$$

where

$$A_{k,r} = -\frac{\binom{2j-2t+r-1}{k}}{\binom{2j-2t+r-1}{j-t}}.$$

The terms with $k=0, 1, \cdots, j-t-2$ are zero modulo $[uv]$ by Lemma 2.1(a). The terms with $k=j-t+1, \cdots, 2j-2t+r-1$ are also zero modulo $[uv]$. To see this, consider the sub-pp.

$$Q_k = u_0^r v_{1+r-k}, \qquad \text{for } j-t=1;$$

and for $j-t>1$,

$$Q_k = u_0u_1 \cdots u_{j-t-2}u_{j-t-1}^r v_1v_2 \cdots v_{j-t-1}v_{2j-2t+r-1-k}.$$

Q_k has signature $(j-t+r-1, j-t)$ and weight $w=(j-t+r-1)(j-t)+(j-t-k)$. Since $j-t<k$, $w<d_1d_2$, and $Q_k \equiv 0[uv]$. The remaining case $k=j-t-1$ gives the lemma.

LEMMA 2.4. *If* $j>0$, *then* $u_0u_1 \cdots u_{j-t-1}u_{j-t}^{t+1}v_1v_2 \cdots v_{j-t}v_{j+1}^{t+1} \equiv B_{t,j}u_0u_1 \cdots u_{j-t-2}u_{j-t-1}^{t+2}v_1v_2 \cdots v_{j-t-1}v_{j+1}^{t+2}[uv]$, $B_{t,j} \neq 0$, *for* $0 \leqq t \leqq j-1$.

PROOF. Apply Lemma 2.3 with $r=1$; then, if $t>0$, apply Lemma 2.3 repeatedly with $r=2, 3, \cdots, t+1$. Note that

$$B_{t,j} = \prod_{r=1}^{t+1} A_{j-t-1,r}$$

and is not zero.

THEOREM 2.5. $u_0u_1 \cdots u_jv_1v_2 \cdots v_{j+1} \equiv C_ju_0^{j+1}v^{j+1}\ [uv]$, $C_j \neq 0$.

PROOF. The theorem follows by a j-fold application of Lemma 2.4, with $C_j = \prod_{t=0}^{j-1} B_{t,j}$ if $j>0$; $C_0=1$.

THEOREM 2.6. $u_0^{j+1}v_{j+1}^{j+1} \equiv (-1)^{j+1}u_{j+1}^{j+1}v_0^{j+1}[uv]$.

PROOF. By Theorem 2.5 and the natural symmetry of $[uv]$, we have

$$(1) \qquad u_0u_1 \cdots u_jv_1v_2 \cdots v_{j+1} \equiv C_ju_0^{j+1}v_{j+1}^{j+1}\ [uv], \qquad C_j \neq 0,$$

$$(2) \qquad u_1u_2 \cdots u_{j+1}v_0v_1 \cdots v_j \equiv C_ju_{j+1}^{j+1}v_0^{j+1}\ [uv].$$

By Theorem 2.2, the conclusion follows.

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FORMS OF ALGEBRAIC GROUPS

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In [4] A. Weil solves the following problem: if V is a variety defined over an overfield K of a groundfield k, among the varieties birationally equivalent to V over K find one which is defined over k. The solution is essentially given by the 1-dimensional Galois cohomology. It was observed by J.-P. Serre that in the case V itself is defined over k the 1-cocycles can be regarded as putting a "twist" into V. In the particular case of simple algebraic groups over finite fields this gives rise to some new finite simple groups.

Let G be an algebraic group defined over a field k and K a Galois extension of k. An algebraic group G' defined over k will be called a k-form of G split by K if there is a rational isomorphism ϕ defined over K between G' and G. Denote by $\mathfrak{g}$ the Galois group of K over k. For $\sigma \in \mathfrak{g}$, $f_\sigma = \phi^\sigma \phi^{-1}$ is an automorphism of G defined over K and for all $\tau, \sigma \in \mathfrak{g}$ we have $f_{\tau\sigma} = f_\sigma^\tau f_\tau$, i.e. f is a 1-cocycle from $\mathfrak{g}$ to $\mathrm{Aut}_K G$, the group of automorphisms of G defined over K.

THEOREM 1. *Let G be a connected algebraic group defined over a field k and K a Galois extension of k with Galois group $\mathfrak{g}$. The distinct k-forms of G (up to k-isomorphism) are in one-to-one correspondence with the elements of $H^1(\mathfrak{g}, \mathrm{Aut}_K G)$.*

PROOF. Let f be a 1-cocycle from $\mathfrak{g}$ to $\mathrm{Aut}_K G$. By Weil's theorem [4, Theorem 1] there exists a variety G' defined over k together

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