A SYMMETRY THEOREM FOR THE DIFFERENTIAL IDEAL $[uv]$

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1. Introduction. Let $F[uv]$ be a Ritt algebra in the indeterminates $u$ and $v$, and let $[uv]$ be the differential ideal generated by the form $X = uv$. If $P = UV$ is a power product (pp.) in $u_i$ and $v_j$ (the subscripts indicate derivatives) and contains no $v_k$, $k < d_1$ ($d_1$ is the degree of $U$), then $P \neq 0 [uv]$. Such pp. are called $\alpha$-terms, and, in particular, a pp. in $u$ alone, a pp. in $v$ alone, and unity are $\alpha$-terms. All other pp. may be reduced modulo $[uv]$ to a linear combination of $\alpha$-terms by H. Levi's reduction process [1; 2]. Levi's methods provide an answer to the question of whether or not a pp. is in the ideal $[uv]$ because a linear combination of $\alpha$-terms is congruent to zero modulo $[uv]$ if and only if all the coefficients are zero. Both the reduction process and the above definitions do not make use of the natural symmetry of the ideal $[uv]$. A pp. $P = UV$ of signature $(d_1, d_2)$ and weight $w = d_1d_2$ is reduced to a multiple of the $\alpha$-term $u_0^d v_0^d$, but, by interchanging the roles of $u$ and $v$, one could reduce $P$ to a multiple of the term $u_0^d v_0^d$. In certain of the problems suggested by J. F. Ritt [3], it would be convenient to know the relationship between $u_0^d v_0^d$ and $u_0^d v_0^d$ so that both types of reductions could be used. The purpose of this note is to exhibit the exact relationship between $u_0^d v_0^d$ and $u_0^d v_0^d$ so that for a pp. of signature $(d, d)$ and weight $w = d^2$, the $u_i$ and $v_i$ may be interchanged.

2. Symmetry theorems. Let $P = UV$ have signature $(d_1, d_2)$ and weight $w = w_1 + w_2$. A theorem of H. Levi states that if $w < d_1d_2$, then $P = 0 [uv]$. Special cases of this theorem are stated for easy reference as

**Lemma 2.1.** (a) If $P_k = u_0 u_1 \cdots u_{k-1} u_2 v_1 v_2 \cdots v_{k+1}$, then $P_k \equiv 0 [uv]$. (b) If $P_k = u_0 u_1 \cdots u_k v_1 v_0 \cdots v_{k+1} v_2$, then $P_k \equiv 0 [uv]$.

**Proof.** (a) The signature of $P_k$ is $(k + 2, k + 1)$ and the weight is $k^2 + 3k + 1$, hence $w < d_1d_2$. The proof of (b) is similar.

**Theorem 2.2.**

$u_0 u_1 \cdots u_j v_1 v_2 \cdots v_{j+1} \equiv (-1)^{j+1} u_1 \cdots u_{j+1} v_0 \cdots v_j [uv]$.

**Proof.** For $j = 0$, $[uv]_1 = u_0 v_1 + u_1 v_0 \equiv 0 [uv]$, hence $u_0 v_1 = - u_1 v_0 [uv]$. Assume that the theorem is true for all values less than $j$. Replacing

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by the other terms in the \((2j+1)\)st derivative of \([uv]\), we have

\[
\begin{align*}
&u_0v_{j+1} \equiv - \sum_{k=0; k \neq j}^{2j+1} \binom{2j+1}{k} v_k u_{2j+1-k} [uv].
\end{align*}
\]

Except for the term \(k = j+1\), each term of the sum is zero modulo \([uv]\) by Lemma 2.1. The induction hypothesis applies to the term \(k = j+1\), and noting that

\[
\frac{\binom{2j+1}{j+1}}{\binom{2j+1}{j}} = 1,
\]

the proof is concluded.

**Lemma 2.3.** If \(j > 0\) and \(0 \leq t \leq j-1\), then

\[
\begin{align*}
&u_0u_1 \cdots u_{j-t-2}v_{j-t-1}v_{j-t}v_{j-t+1}v_{j-t+r}v_{j+1} \\
&\equiv - \binom{2j-2t+r-1}{j-1} \cdot \binom{2j-2t+r-1}{j-1} \\
&\times u_0u_1 \cdots u_{j-t-2}v_{j-1}v_{j-t}v_{j-t+r}v_{j+1} [uv]
\end{align*}
\]

for \(0 < r \leq t+1\).

**Proof.** Replace \(u_{j-t}v_{j-t+r-1}\) by the other terms in the \((2j-2t+r-1)\)th derivative of \([uv]\) and get the congruence

\[
\begin{align*}
&u_0u_1 \cdots u_{j-t-2}v_{j-t-1}v_{j-t}v_{j-t+1}v_{j-t+r}v_{j+1} \\
&\equiv - \sum_{k=0; k \neq j-1}^{2j-2t+r-1} A_{k,r} u_k v_{2j-2t+r-1-k} [uv],
\end{align*}
\]

where
The terms with \( k = 0, 1, \cdots, j - t - 2 \) are zero modulo \([uv]\) by Lemma 2.1(a). The terms with \( k = j - t + 1, \cdots, 2j - 2t + r - 1 \) are also zero modulo \([uv]\). To see this, consider the sub-pp.

\[
A_{k, r} = -\binom{2j - 2t + r - 1}{k} \binom{2j - 2t + r - 1}{j - t}.
\]

For \( j - t = 1 \),

\[
Q_k = u_0 v_{1+r-k},
\]

and for \( j - t > 1 \),

\[
Q_k = u_0 u_1 \cdots v_{j-t+2-k} v_{j-t-1} v_{1} v_{2} \cdots v_{j-t-1} v_{2j-2t+r-1-k}.
\]

\( Q_k \) has signature \((j-t+r-1, j-t)\) and weight \( w = (j-t+r-1)(j-t) + (j-t-k) \). Since \( j-t < k \), \( w < d_1 d_2 \), and \( Q_k = 0[uv] \). The remaining case \( k = j - t - 1 \) gives the lemma.

**Lemma 2.4.** If \( j > 0 \), then \( u_0 u_1 \cdots u_{j-t-1} u_{j-t} v_1 v_2 \cdots v_{j-t-1} v_{j-t} v_{j+1} = B_{t,j} u_0 u_1 \cdots u_{j-t-2} u_{j-t-1} v_1 v_2 \cdots v_{j-t-1} v_{j+1} [uv] \), \( B_{t,j} \neq 0 \), for \( 0 \leq t \leq j-1 \).

**Proof.** Apply Lemma 2.3 with \( r = 1 \); then, if \( t > 0 \), apply Lemma 2.3 repeatedly with \( r = 2, 3, \cdots, t+1 \). Note that \( B_{t,j} = \prod_{r=1}^{t+1} A_{j-t-1,r} \) and is not zero.

**Theorem 2.5.** \( u_0 u_1 \cdots u_{j-t-1} u_{j-t} v_1 v_2 \cdots v_{j+1} = C_j u_0 v_{j+1} [uv] \), \( C_j \neq 0 \).

**Proof.** The theorem follows by a \( j \)-fold application of Lemma 2.4, with \( C_j = \prod_{i=0}^{j-1} B_{t,i} \) if \( j > 0 \); \( C_0 = 1 \).

**Theorem 2.6.** \( u_0 v_{j+1} = (-1)^{j+1} u_{j+1} v_0 v_{j+1} [uv] \).

**Proof.** By Theorem 2.5 and the natural symmetry of \([uv]\), we have

1. \( u_0 u_1 \cdots u_{j-t-1} u_{j-t} v_1 v_2 \cdots v_{j+1} = C_j u_0 v_{j+1} [uv] \), \( C_j \neq 0 \),
2. \( u_1 u_2 \cdots u_{j+1} v_0 v_1 \cdots v_j = C_j u_{j+1} v_0 [uv] \).

By Theorem 2.2, the conclusion follows.

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**FORMS OF ALGEBRAIC GROUPS**

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In [4] A. Weil solves the following problem: if \( V \) is a variety defined over an overfield \( K \) of a groundfield \( k \), among the varieties birationally equivalent to \( V \) over \( K \) find one which is defined over \( k \). The solution is essentially given by the 1-dimensional Galois cohomology. It was observed by J.-P. Serre that in the case \( V \) itself is defined over \( k \) the 1-cocycles can be regarded as putting a “twist” into \( V \). In the particular case of simple algebraic groups over finite fields this gives rise to some new finite simple groups.

Let \( G \) be an algebraic group defined over a field \( k \) and \( K \) a Galois extension of \( k \). An algebraic group \( G' \) defined over \( k \) will be called a \( k \)-form of \( G \) split by \( K \) if there is a rational isomorphism \( \phi \) defined over \( K \) between \( G' \) and \( G \). Denote by \( g \) the Galois group of \( K \) over \( k \).

**Theorem 1.** Let \( G \) be a connected algebraic group defined over a field \( k \) and \( K \) a Galois extension of \( k \) with Galois group \( g \). The distinct \( k \)-forms of \( G \) (up to \( k \)-isomorphism) are in one-to-one correspondence with the elements of \( H^1(g, \text{Aut}_K G) \).

**Proof.** Let \( f \) be a 1-cocycle from \( g \) to \( \text{Aut}_K G \). By Weil’s theorem [4, Theorem 1] there exists a variety \( G' \) defined over \( k \) together with a rational isomorphism \( \phi \) of \( G' \) with \( G \) which is defined over \( K \).

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