OBSERVATIONS ON A PAPER BY ROSENBLUM

S. CATER

M. Rosenblum in [2] presented a most ingenious proof of the Fuglede and Putnam Theorems by means of entire vector valued functions [1, p. 59]. We will demonstrate that some curious properties of bounded Hilbert space operators can be derived from Rosenblum's argument and similar arguments.

Throughout this text we mean by an "operator" a bounded linear transformation of a Hilbert space into itself. Given an operator A we mean by "exp A" the uniform limit of the series $I+A+A^2/2!+A^3/3!+A^4/4!+\cdots$. We let A^* denote the adjoint of the operator A, and let z^* denote the complex conjugate of the complex number z. A "normal" operator is an operator which commutes with its adjoint.

A critical fact in the Rosenblum proof is that given a normal operator A and any complex number z, $\exp(izA)\exp(iz^*A^*)=\exp(izA+iz^*A^*)=\exp(iz^*A^*)\exp(izA)$, and this operator is unitary because $i(zA+z^*A^*)$ is skew hermitian. Our first result states, among other things, that the converse is true; if the above equations hold for a fixed operator A and all complex numbers z, then A is normal.

THEOREM 1. *The following conditions are equivalent for an operator A.*

(1) $A^*A=AA^*$,

(2) $\exp(zA)\exp(-zA^*)$ *is unitary for all real z,*

(3) $\exp(zA)\exp(-z^*A^*)$ *and* $\exp(-z^*A^*)\exp(zA)$ *are uniformly bounded in complex z.*

PROOF. It is trivial that (1) $\Rightarrow$ (2), (3). Assume (2). Put $U=\exp(zA)\exp(-zA^*)$, for some real z. $\exp(-zA)\exp(zA^*)=U^*=U^{-1}=\exp(zA^*)\exp(-zA)$. Thus $\exp(-zA)\exp(zA^*)=\exp(zA^*)\exp(-zA)$, all real z. Differentiating twice with respect to z and setting $z=0$, we have $A^*A=AA^*$. (2)$\Rightarrow$(1). Now assume (3). For all integers n, $A\cdot A^n=A^n\cdot A$. For any complex z, $A\exp(z^*A)=\exp(z^*A)A$, $A=\exp(z^*A)A\exp(-z^*A)$. But $\exp(-zA^*)A\exp(zA^*)=\exp(-zA^*)\exp(z^*A)A\exp(-z^*A)\exp(zA^*)$ is an entire function which, by (3), is bounded in the complex plane. By Liouville's Theorem this operator is constant, and clearly $\exp(-zA^*)A\exp(zA^*)=\exp(0)A\exp(0)=A$. Differentiating $A\exp(zA^*)=\exp(zA^*)A$ with respect to z and setting $z=0$ we get $AA^*=A^*A$. Hence (3)$\Rightarrow$(1). This completes the proof.

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Observe that if A and B are operators then $AB=BA$ iff $\exp(zA)\exp(zB)=\exp(zB)\exp(zA)$ for all real z. To establish this, differentiate the latter equation twice with respect to z and set $z=0$.

The next theorem has several significant special cases. When $B=A^*$, (1)–(3) state that A is hermitian (or self adjoint). When $B=-A^*$, (1)–(3) state that A is skew hermitian. When $B=A^*A$, (1)–(3) state that A is a projection; this is due to the fact that if $A=A^*A$, then (taking adjoints) $A^*=A^*A=A$ and (substituting A for A^*) $A=A^2$. When $A=T^*T$ and $B=T^*TT^*T$, (1)–(3) state that T is a partial isometry. When $A=T^*$ and $B=T^{-1}$, (1)–(3) state that T is unitary.

THEOREM 2. *The following are equivalent for two operators A and B.*
(1) $A=B$,
(2) $\exp(zA)\exp(z^*B)=\exp(zB)\exp(z^*A)$, *all complex z,*
(3) $\exp(zA)\exp(-zB)=\exp(zB)\exp(-zA)$, *all real z.*

PROOF. Clearly (1)$\Rightarrow$(2). Assume (2). $\exp(zA)\exp(-zB)$ and $\exp(zB)\exp(-zA)$ are entire functions which coincide for z purely imaginary, hence for z complex. (2)$\Rightarrow$(3). Now assume (3). Differentiating (3) with respect to z and setting $z=0$, we have $A-B=B-A$ and $A=B$. (3)$\Rightarrow$(1).

Next we find necessary and sufficient conditions that an operator be the zero operator.

THEOREM 3. *For an operator A the following are equivalent.*
(1) $A=0$,
(2) $\exp(zA)=\exp(z^*A^*)$, *all complex z,*
(3) $\exp(zA)\exp(zA^*)=\exp(zA^*)\exp(-zA)$, *all real z.*

PROOF. Obviously (1)$\Rightarrow$(2), (3). Assume (2). For all purely imaginary z, $\exp(zA)=\exp(-zA^*)$. Differentiating with respect to z and setting $z=0$, we get $A=-A^*$. Then (2) reduces to $\exp(zA)=\exp(-z^*A)$, and for real z, $\exp(2zA)=I$. Differentiating with respect to z and setting $z=0$ again we get $A=0$. (2)$\Rightarrow$(1). Now assume (3). Differentiating (3) twice and setting $z=0$ we get $A^*A=-AA^*$ and $A^*A+AA^*=0$. Hence $A^*A=0$ and $A=0$. (3)$\Rightarrow$(1).

REFERENCES

1. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publications, vol. 31, 1957.

2. M. Rosenblum, *On a theorem of Fuglede and Putnam*, J. London Math. Soc. vol. 33 (1958) pp. 376–377.

UNIVERSITY OF OREGON