ON THE EXISTENCE OF CROSS SECTIONS IN
LOCALLY FLAT BUNDLES

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1. A principal bundle $\mathfrak{B}(X, G)$ with base space $X$ and group $G$ is called “locally flat” if the structural group $G$ can be reduced to a totally disconnected subgroup $G_1 \subset G$, or in other words: if there exists a totally disconnected subgroup $G_1 \subset G$, a principal bundle $\mathfrak{B}_1(X, G_1)$ and an injection $j: \mathfrak{B}_1 \to \mathfrak{B}$ \cite{1}. For instance, if a differentiable bundle $\mathfrak{B}$ admits an infinitesimal connection whose curvature form is identically zero, then $\mathfrak{B}$ is locally flat \cite[Reduction theorem, p. 37]{1}.

We are mainly interested in locally flat bundles with arcwise connected structural group $G$. The aim of this note is to give necessary and sufficient conditions for locally flat bundles to have a cross section.

2. We have to make use of certain more or less obvious properties of locally flat bundles which were discussed in a previous paper. As all bundles in the sequel will be principal bundles we shall omit the word “principal”; we assume all base spaces $X$ of bundles to be arcwise connected.

**Definition.** A reduction of a bundle $\mathfrak{B}(X, G)$ is the collection of a subgroup $G_1 \subset G$, bundle $\mathfrak{B}_1(X, G_1)$ and injection $j: \mathfrak{B}_1 \to \mathfrak{B}$. Suppose $G$ can be reduced to a totally disconnected subgroup $G_1 \subset G$. Corresponding to points of reference $x_0 \in X$ and $b_1 \in B_1$, where $B_1$ is the bundle space of $\mathfrak{B}_1$, there exists the characteristic homomorphism $\chi: \pi_1(X) \to G_1$ \cite[p. 61]{2}. We say that the reduction is “irreducible” if $\chi$ is onto.

**Definition.** Let $j: \mathfrak{B}_1(X, G_1) \to \mathfrak{B}(X, G)$ be a reduction, $G_1$ totally disconnected. $\chi: \pi_1(X) \to G_1$ is called a “characteristic homomorphism of the reduction,” $H = \ker(\chi)$ the “kernel of the reduction.”

3. From now on we assume $X$ to be arcwise connected, arcwise locally connected and semi locally $1$-connected. This will make sure the existence of bundles over $X$ with totally disconnected group and prescribed characteristic homomorphism. In particular, to every invariant subgroup $N \subset \pi_1(X)$ belongs a regular covering $p_2: \tilde{X} \to X$ such that $\pi_1(\tilde{X}) \approx N$. If we put $P = \pi_1(X)/N$ then $\tilde{X}$ is the space of a bundle $\mathfrak{B}_2(X, P)$ with discrete group $P$, and the right action of $p \in P$ on $\tilde{x} \in \tilde{X}$—denoted by $\tilde{x} \cdot p$—is a “deckbewegung.”

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Theorem 1. Let $\mathfrak{B}(X, G)$ be locally flat, $\chi: \pi_1(X) \to G_1 \subset G$ a characteristic homomorphism and $H$ the kernel of a reduction $j: \mathfrak{B}_1(X, G_1) \to \mathfrak{B}(X, G)$. Let $N$ be an invariant subgroup of $\pi_1(X)$ such that $N \subset H$. $\chi$ induces a homomorphism $P \to G_1$ again denoted by $\chi$. Let $\tilde{X}$ be the covering space of $X$ corresponding to $N$. Then, $\mathfrak{B}(X, G)$ has a cross section if and only if there exists a map $f: \tilde{X} \to G$ such that

$$f(\tilde{x} \cdot \vec{p}) = f(\tilde{x}) \chi(\vec{p}) \quad \text{for all } \tilde{x} \in \tilde{X} \text{ and } \vec{p} \in P.$$ 

Theorem 2. Suppose a homomorphism $\chi: \pi_1(X) \to G_1 \subset G$ is given, and $G_1$ is totally disconnected. Then there exist bundles $\mathfrak{B}(X, G)$, $\mathfrak{B}_1(X, G_1)$ and injection $j: \mathfrak{B}_1 \to \mathfrak{B}$ such that $\chi$ is a characteristic homomorphism of this reduction. Furthermore, any two such bundles $\mathfrak{B}(X, G)$ are equivalent.

For the proof of Theorems 1 and 2 we refer to [3].

4. Definition. Let $T^n$ denote the $n$-dimensional torus, $n \geq 0$; $T^0$ consists of one point. Let $G$ be a topological group and $G_1$ a subgroup $\subset G$. We say that $G_1$ is $n$-flat in $G$ if every bundle $\mathfrak{B}(T^n, G)$ over $T^n$ whose structural group $G$ can be reduced to $G_1$, has a cross section.

Clearly, $G_1 = \{e\}$ is $n$-flat in $G$ for every $n$. If $G$ is arcwise connected, every subgroup of $G$ is 1-flat in $G$. If $G_1$ is $n$-flat in $G$, then $G_1$ is $p$-flat in $G$ for every $0 \leq p \leq n$. As will be shown later, $n$-flat subgroups which are totally disconnected are characterized by topological properties.

Theorem 3. Suppose $X$ is, in addition to being connected in the sense of §3, a normal space with the property that every covering of $X$ by open sets is reducible to a countable covering. Let $\mathfrak{B}(X, G)$ be locally flat, $j: \mathfrak{B}_1(X, G_1) \to \mathfrak{B}(X, G)$ a reduction, and $H$ the kernel of the reduction. Suppose there exists an invariant subgroup $N \subset \pi_1(X)$ such that $N \subset H$ and $P = \pi_1(X)/N$ is a finitely generated free abelian group whose dimension is $n$. Then $\mathfrak{B}$ has a cross section if $G_1$ is $n$-flat in $G$.

5. The proof of Theorem 3 will be preceded by two lemmas.

Lemma 1. Let $X$ and $N \subset \pi_1(X)$ be the same as in Theorem 3. Let $p_2: \tilde{X} \to X$ be the covering corresponding to $N$, $\{p_1, \ldots, p_n\}$ a basis of $P = \pi_1(X)/N$, $\{e_1, \ldots, e_n\}$ a basis of Euclidean space $E^n$. The correspondence $p_i \mapsto e_i$ sets up an isomorphism $\psi: P \to Z^n$, where $Z^n$ is the vector group generated by $\{e_1, \ldots, e_n\}$. Then, there exists a map $t: \tilde{X} \to E^n$ such that $t(\tilde{x} \cdot \vec{p}) = t(\tilde{x}) + \psi(\vec{p})$.

The composition of the natural homomorphism $\phi: \pi_1(X) \to P$ and $\psi$ is a homomorphism $\psi_\phi: \pi_1(X) \to E^n \subset E^n$, $E^n$ being the additive group.
of $E^n$. By Theorem 2 there exist bundles $\mathcal{B}(X, E^n)$, $\mathcal{B}_1(X, Z^n)$ and injection $j_1: \mathcal{B}_1 \to \mathcal{B}$, such that $\psi_1$ is a characteristic homomorphism of this reduction. The topological space of $E^n$ is solid and $X$ is such that every bundle over $X$ with a solid fibre has a cross section [2, p. 55]. Hence, $\mathcal{B}(X, E^n)$ has a cross section and the map $t$ of the lemma is the map $f$ given by Theorem 1, the right hand side of (3.1) written additively and $\chi$ replaced by $\psi$.

Lemma 2. The situation being as in Theorem 3 and Lemma 1, let $T^n$ be the $n$-dimensional torus and $\{u_1, \ldots, u_n\}$ a basis of $\pi_1(T^n)$. $E^n$ is the universal covering space of $T^n$, and we can assume that the lifting of curves in the class $u_i$ provides curves with initial point $O \in E^n$ and endpoint $e_i \in E^n$. This induces an isomorphism $\pi_1(T^n) \approx Z^n$, with respect to which $\chi_1 = \chi_t^{-1}$ can also be interpreted as a homomorphism $\pi_1(T^n) \to G_1$. By Theorem 2 there exist bundles $\mathcal{B}(T^n, G)$, $\mathcal{B}_1(T^n, G_1)$ and injection $j_1: \mathcal{B}_1 \to \mathcal{B}$ such that $\chi_1$ is a characteristic homomorphism of this reduction.

Suppose now, $G_1$ is $n$-flat in $G$. Then the bundle $\mathcal{B}(T^n, G)$ has a cross section which, on the other hand, means that there exists a map $f_1: E^n \to G$ such that $f_1(v+z) = f_1(v)\chi_1(z)$ for all $v \in E^n$ and $z \in Z^n$. The composition of this $f_1$ with $t$ given by Lemma 1 is a map $f = f_1t: \tilde{X} \to G$ which has the property (3.1). Hence $\mathcal{B}(X, G)$ has a cross section and Theorem 3 is proved.

6. Theorem 4. Let the notations be as before. Assume the reduction in Theorem 3 to be irreducible. Suppose there exists a map $s: E^n \to \tilde{X}$ such that $s(v+z) = s(v)\psi_t^{-1}(z)$ for all $v \in E^n$ and $z \in Z^n$. Then $\mathcal{B}(X, G)$ has a cross section if and only if $G_1$ is $n$-flat in $G$.

The sufficiency is given by Theorem 3. Let $\mathcal{B}(X, G)$ have a cross section. There exists a map $f: \tilde{X} \to G$ with the property (3.1). The composition of $f$ with $s$ is a map $f_1 = fs: E^n \to G$ such that $f_1(v+z) = f_1(v)\chi_1(z)$ for all $v \in E^n$ and $z \in Z^n$. This means that $\mathcal{B}(T^n, G)$ has a cross section (Lemma 2 and Theorem 1). The proof of Theorem 4 will be complete after the correctness of the following has been shown.

Lemma 3. A totally disconnected subgroup $G_1 \subset G$ is $n$-flat in $G$ if there exist bundles $\mathcal{B}(T^n, G)$, $\mathcal{B}_1(T^n, G_1)$ and an irreducible reduction $j_1: \mathcal{B}_1 \to \mathcal{B}$ (we mean by this that all characteristic homomorphisms $\chi_1: \pi_1(T^n) \to G$ are onto) such that $\mathcal{B}(T^n, G)$ has a cross section.

The lemma will be proved later. We are going to explain now the notion of $n$-flatness in detail.

7. In the following, $G_1 \subset G$ will always be assumed to be totally
disconnected. Let \( \{ h_1, \cdots, h_n \} \) be a commutative set of \( n \) elements in \( G_1 \). We mean by this a set of \( n \) elements \( h_i \in G_1 \) such that \( h_j h_k = h_k h_j \). Let \( \{ e_1, \cdots, e_n \} \) be a basis of Euclidean space \( E^n \).

**Lemma 4.** If \( G_1 \) is \( n \)-flat in \( G(n \geq 1) \), then there exists a map \( f: E^n \to G \) such that

\[
(7.1) \quad f(v + e_i) = f(v) h_i \quad \text{for all } v \in E^n \text{ and } i = 1, \cdots, n.
\]

If \( \{ p_1, \cdots, p_n \} \) is a basis of \( \pi_1(T^n) \) we can assume that the lifting of curves in the class \( p_i \) provides curves from \( 0 \in E^n \) to \( e_i \in E^n \). The correspondence \( p_i \to h_i \) generates a homomorphism \( \chi_i: \pi_1(T^n) \to G_1 \). By Theorem 2 there exists a bundle \( \mathcal{B}(T^n, G) \) and a reduction of \( G \) to \( G_1 \) with characteristic homomorphism \( \chi_i \). If \( G_1 \) is \( n \)-flat in \( G \), \( \mathcal{B}(T^n, G) \) has a cross section, and \( f \) in Lemma 4 is the map \( g \) given by Theorem 1.

Let now \( T^n(\mathbb{Z}) \) be the \( n \)-dimensional torus and \( G \) a topological group. Let \( t_0 \in T^n \) be a point of reference and \( F^n \) the set of continuous mappings \( f: (T^n, t_0) \to (G, e) \). If \( f_1 \in F^n \), \( f_2 \in F^n \), then \( f_1 \cdot f_2 \) denotes the mapping \( f_1 \cdot f_2(t) = f_1(t) \cdot f_2(t), \quad t \in T^n \), and \( f^{-1} \) the composition of \( f_1: T^n \to G \) with the map \( g \to g^{-1} \) in \( G \). Let \( \tau_n(G) \) denote the set of homotopy classes of mappings \( f \in F \). \( \tau_n(G) \) is a group, the product of two classes \( u_1 \) and \( u_2 \) being defined as the class of products \( f_1 \cdot f_2 \), \( f_1 \in u_1, f_2 \in u_2 \). Clearly, \( \tau_1(G) = \pi_1(G) = \text{fundamental group of } G \).

Suppose then \( \{ h_1, \cdots, h_n \} \) is a commutative set of elements of \( G_1 \). If \( G_1 \) is \( n \)-flat in \( G \) and \( \{ e_1, \cdots, e_n \} \) is a basis of \( E^n \) there exists a map \( f: E^n \to G \) with property (7.1). We may even assume

\[
(7.2) \quad f(0) = e = \text{identity in } G,
\]

because, if \( f \) satisfies (7.1) and \( a = f(0) \) then \( g(v) = a^{-1} \cdot f(v) \) satisfies both (7.1) and (7.2). This being said, let \( h_{n+1} \in G_1 \) be such that

\[
h_i h_{n+1} = h_{n+1} h_i, \quad i = 1, \cdots, n.
\]

Because of the commutativity, \( h_{n+1} \cdot f \cdot h_{n+1} : E^n \to G \) will again have properties (7.1) and (7.2). If two mappings \( f_1 : E^n \to G \) and \( f_2 : E^n \to G \) have properties (7.1) and (7.2), then \( f_1 \cdot f_2^{-1} : E^n \to G \) induces a map \( \tilde{g}: (T^n, t_0) \to (G, e) \) where \( t_0 = p_2(0) \) and \( p_2 : E^n \to T^n \) is the covering, such that \( f_1 \cdot f_2^{-1} = \tilde{g} p_2 \). Hence \( h_{n+1} \cdot f \cdot h_{n+1} \cdot f^{-1} \) induces a map \( \tilde{f}_1 : (T^n, t_0) \to (G, e) \) which lies in a homotopy class \( u \in \tau_n(G) \). If the elements \( h_1, \cdots, h_{n+1} \) are kept fixed, \( u \) still might depend on \( f \). Let \( f_1 : E^n \to G \) be another map with properties (7.1) and (7.2), and \( \tilde{f}_1 \) be the map \( \in F^n \) induced by \( h_{n+1}^{-1} \cdot f_1 \cdot h_{n+1} \cdot f_1^{-1} \). Applying the reasoning above, \( f_1 \cdot f_1^{-1} = g \) induces a map \( \tilde{g} \in F^n \). From \( h_{n+1}^{-1} \cdot f_1 \cdot f_1^{-1} = h_{n+1}^{-1} \cdot g \cdot h_{n+1}^{-1} \cdot f_1 \cdot f_1^{-1} \cdot g \) follows \( \tilde{f}_1 = h_{n+1}^{-1} \cdot \tilde{g} \cdot h_{n+1}^{-1} \cdot \tilde{f} \cdot \tilde{g}^{-1} \). As \( h_{n+1}^{-1} \cdot \tilde{g} \cdot h_{n+1} \) is homotopic to \( \tilde{g} \) in \( F^n \) one gets \( u_1 = u_0 \cdot u \cdot u_0^{-1} \), where \( u_1, u_0 \) is the homotopy class of \( \tilde{f}_1, \tilde{g} \) resp. In short, we have the following result.
Lemma 5. Let $G_1$ be $n$-flat in $G$, $n \geq 1$. The construction above assigns to every commutative set of $(n+1)$ elements $h_i \in G_1$, $i = 1, \ldots, (n+1)$, an equivalence class $\psi_n(h_1, \ldots, h_{n+1})$ under inner automorphisms of elements $\in \tau_n(G)$.

8. We adopt the convention that, if $\psi$ denotes an equivalence class under inner automorphisms of elements of a group $K$, $\psi = 0$ means it contains the identity $e \in K$.

Theorem 5. A totally disconnected group $G_1 \subset G$ is $(n+1)$-flat in $G(n \geq 1)$, if and only if it is $n$-flat in $G$ and $\psi_n(h_1, \ldots, h_{n+1}) = 0$ for every commutative set of elements in $G_1$. The statement: $G_1$ is $1$-flat in $G$, is equivalent to: $G_1$ lies in the arcwise connected component of $e \in G$.

The second part of Theorem 5 is obvious. As for the first part, let us assume that $G_1$ is $n$-flat in $G(n \geq 1)$ and $\psi_n(h_1, \ldots, h_{n+1}) = 0$. Let $\mathfrak{B}(T^{n+1}, G)$, $\mathfrak{B}_1(T^{n+1}, G_1)$ be bundles, $j: \mathfrak{B}_1 \rightarrow \mathfrak{B}$ an injection and $\chi: \pi_1(T^{n+1}) \rightarrow G_1$ a characteristic homomorphism of this reduction. Let $\rho_2: E^{n+1} \rightarrow T^{n+1}$ be the universal covering, $\{p_1, \ldots, p_{n+1}\}$ a basis of $\pi_1(T^{n+1})$, $\{e_1, \ldots, e_{n+1}\}$ a basis of $E^{n+1}$ such that the lifting of closed curves in the class $p_i \in \pi_1(T^{n+1}, t_0)$ provides curves in $E^{n+1}$ from 0 to $e_i$, $t_0 = p_2(0)$. The set of elements $h_i = \chi(p_i) \in G_1$, $i = 1, \ldots, (n+1)$, is certainly commutative. As $G_1$ is $n$-flat in $G$ there exists by Lemma 4 a map $f_i: E^n \rightarrow G$ such that

\[(7.1) \quad f_i(v + e_i) = f_i(v)h_i, \quad \text{for all } v \in E^n \text{ and } i = 1, \ldots, n,
\]

holds. Here $E^n$ denotes the space spanned by the vectors $e_1, \ldots, e_n$. We may assume $f_1(0) = e \in G$ (7.2). Denote by $g: E^n \rightarrow G$ the map $h_0^{-1} \cdot f_1 \cdot h_n^{-1} \cdot f_1^{-1}$, by $\hat{g}: T^n \rightarrow G$ the induced map such that $g = \hat{g} \rho_2$, where $\rho_2: E^n \rightarrow T^n$ is the covering map. By the assumption $\psi_n(h_1, \ldots, h_{n+1}) = 0$ there exists a homotopy $\tilde{h}: (I \times T^n, t_0) \rightarrow (G, e)$ such that $\tilde{h}(1, t) = \hat{g}(t)$ and $\tilde{h}(0, t) = e$ for $t \in T^n$. Then $\tilde{h}$ defined by $h(\rho, v) = \tilde{h}(\rho, \rho^2(0))$ for $\rho \in I$ and $v \in E^n$, is a homotopy of $g$ such that

\[(8.1) \quad h(1, v) = g(v), \quad h(0, v) = e = h(\rho, 0), \quad h(\rho, v + e_i) = h(\rho, v), \quad i = 1, \ldots, n.
\]

Let now $w: I \rightarrow G$ be a curve connecting $e = w(0)$ and $w(1) = h_{n+1}$. Then $f^*$ defined by $f^*(\rho, v) = w(\rho)h(\rho, v)f_1(v)$ is a mapping $I \times E^n \rightarrow G$ which has the following properties

\[(8.2) \quad f^*(0, v) = f_1(v), \quad f^*(\rho, v + e_i) = f^*(\rho, v)h_i \quad (i = 1, \ldots, n),
\]

\[f^*(1, v) = f^*(0, v)h_{n+1}.
\]

Any vector $w \in E^{n+1}$ can be written as $\rho e_{n+1} + v$, $v \in E^n$. Let $m$ be an
integer and $E_m$ the set $\{w=\rho e_{n+1}+v\mid m\leq\rho\leq(m+1)\}$. We define mappings $f_m: E_m\to G$ by $f_m(w)=f^*(\rho-m,v)h_m$. Because of (8.2) one has $f_m(w)=f_{m+1}(w)$ if $w\in E_m\cap E_{m+1}$, thus the collection $\{f_m\}$ determines one map $f: E^{n+1}\to G$ with the property $f(w+e_i)=f(w)h_i$, $i=1,\ldots,(n+1)$, where $h_i=\chi(e_i)$. But the existence of such a map asserts the existence of a cross section in $\mathcal{S}(T^{n+1},G)$ (Theorem 1). Hence $G_1$ is $(n+1)$-flat in $G$.

Conversely suppose $G_1$ to be $(n+1)$-flat in $G(n\geq 1)$. As was pointed out in connection with the definition of $n$-flatness, $(n+1)$-flatness induces $n$-flatness. Let $\{h_1,\ldots,h_{n+1}\}$ be a commutative set of elements in $G_1$. Using the same notations as before, the correspondence $p_i\in\pi_i(T^{n+1})\to h_i$ generates a homomorphism $\chi: \pi_1(T^{n+1})\to G_1$ into. By Theorem 2, $\chi$ is a characteristic homomorphism of a certain reduct $j: \mathcal{B}_i(T^{n+1}, G_1)\to \mathcal{B}(T^{n+1}, G)$ where $\mathcal{B}(T^{n+1}, G)$ has a cross section because of the assumed $(n+1)$-flatness of $G_1$ in $G$. By Theorem 1 there exists a map $f: E^{n+1}\to G$ such that $f(w+e_i)=f(w)h_i$ for $w\in E^{n+1}$ and $i=1,\ldots,(n+1)$. Denote by $f_1$ the restriction $f|E^n$. We may assume $f_1(0)=e$ (7.2). As $w=\rho e_{n+1}+v$, $v\in E^n$, we can write $f(w)=f_2(v)$. Denote by $u(\rho)$ the expression $f_2(0)=f(\rho e_{n+1})$. Then $h$ defined by $h(\rho, v)=u^{-1}(\rho)f_2(v)\bar{f}(v)\bar{f}^{-1}(v)$ is continuous in $E^{n+1}$ and represents for $0\leq\rho\leq 1$ a homotopy of $h(1,v)=h_{n+1}^{-1}f_2(v)\bar{h}_n\bar{f}^{-1}(v)=u^{-1}(\rho)f_2(v)\bar{f}^{-1}(v)$ in $h(0,v)=e$. As for all $\rho$ $u^{-1}(\rho)f_2(v+e_i)\bar{f}^{-1}(v+e_i)=u^{-1}(\rho)f_2(v)\bar{f}^{-1}(v)$, $h$ will induce a homotopy $h: (I\times T^n, t_0)\to (G, e)$ of the map $g: (T^n, t_0)\to (G, e)$ induced by $h_{n+1}^{-1}f_2\bar{h}_n\bar{f}^{-1}$ in the map $T^n\to e$. Therefore $\psi_n(h_1,\ldots,h_{n+1})=0$, and the proof of Theorem 5 is complete.

9. There still remains to prove Lemma 3. The situation is the following: two bundles $\mathcal{B}(T^n, G)$, $\mathcal{B}_i(T^n, G_1)$ together with a reduct $j: \mathcal{B}_i\to \mathcal{B}$ are given such that $G_1\subset G$ is totally disconnected, $\mathcal{B}(T^n, G)$ has a cross section and, if $\chi$ denotes a characteristic homomorphism, $\chi: \pi_1(T^n)\to G_1$ is onto. Using the same notations as before, there exists a map $f: E^n\to G$ such that $f(v+e)\chi(\bar{v})$ for $v\in E^n$ and $z\in Z^n$ (Theorem 1) and $f(0)=e$ (7.2). $Z^n$ is the vector group generated by $\{e_1,\ldots,e_n\}$ and as such isomorphic to $\pi_1(T^n)$. Any $g_i\in G_i$ is the image under $\chi$ of a certain $z\in Z^n$. Let $g_1=\chi(z)$, then $u(\rho)=f(\rho z)$ is a curve in $G$ from $e$ to $g_1$. This shows that $G_1$ is a curvewise connected component of $G$ and hence is 1-flat in $G$ (Theorem 5). Suppose $G_1$ is already proved to be $p$-flat in $G$, $1\leq p<n$. Let $\{h_1,\ldots,h_{p+1}\}$ be an ordered set of elements in $G_1$; note that $G_1$ is abelian. Let $v_i\in Z^n$ be such that $\chi(v_i)=h_i$. If $\{e_1,\ldots,e_{p+1}\}$ is a basis of $E^{p+1}$ then the correspondence $e_i\to v_i$ generates a linear map $\phi$ of $E^{p+1}$ onto the vector-space spanned by $\{v_1,\ldots,v_{p+1}\}$. The composition $f\phi: E^{p+1}\to G$ has
the property \( f\phi(w + e_i) = f\phi(w)h_i \) for \( w \in E^{p+1} \) and \( i = 1, \ldots, (p+1) \).

But now we have the same situation as in the second part of the proof of Theorem 5. The reasoning applied there leads to the conclusion \( \psi_p(h_1, \ldots, h_{p+1}) = 0 \). It follows from Theorem 5 that \( G_1 \) is \((p+1)\)-flat in \( G \).

10. **Corollary 1.** Every totally disconnected subgroup \( G_1 \) of an arcwise connected abelian group \( G \) is \( n \)-flat in \( G \) for every \( n \).

This follows directly from Theorem 5. One just has to observe that elements of \( \psi_n(h_1, \ldots, h_{n+1}) \) are homotopy classes of mappings of the form \( h_{n+1}^{-1} \cdot f \cdot h_{n+1} \cdot f^{-1} \).

**Corollary 2.** Let \( X \) be a topological space as in Theorem 3. If \( \pi_1(X) \) is a finitely generated free abelian group, then every locally flat bundle with base space \( X \) and arcwise connected abelian group \( G \) has a cross section.

This is a consequence of Theorem 3 and Corollary 1.

11. We wish to give an example of a locally flat bundle without cross section. Let \( R^3 \) denote the group of rotations in Euclidean space \( E^3 \) and \( \{e_1, e_2, e_3\} \) be an orthogonal basis in \( E^3 \). Define \( h_1 \) and \( h_2 \in R^3 \) by \( h_1(e_i) = -e_i, i = 1, 2, h_1(e_3) = e_3, h_2(e_i) = e_i, h_2(e_3) = -e_3 \) for \( k = 2, 3 \). Clearly \( h_1h_2 = h_2h_1 \). The set of rotations in \( R^3 \) leaving \( e_3 \) fixed may be identified with \( R^2 \). If \( g \in R^2 \), then \( h_3^{-1}gh_3 = g^{-1} \), especially if \( f: I \rightarrow R^2 \) is a curve from \( e \) to \( h_1, h_3^{-1}fh_3 = f^{-1} \). As \( h_1^2 = e, u = h_3^{-1}fh_3f^{-1} = f^{-2} \) is a closed curve in \( R^2 \), and we can choose \( f \) such that the homotopy class of \( u \) in \( R^2 \) generates \( \pi_1(R^2) \). But then \( u \) cannot be homotopic to zero in \( R^3 \) either, which means \( \psi_1(h_1, h_2) \neq 0 \). Hence the discrete group \( G_1 = \{e, h_1, h_2, h_3\} \) is not 2-flat in \( R^3 \). If \( T^2 \) is now the two-dimensional torus we define \( \chi: \pi_1(T^2) \rightarrow G_1 \) by \( \chi(p_i) = h_i, i = 1, 2, \) where \( \{p_1, p_2\} \) in a basis of \( \pi_1(T^2) \). Note that \( \chi \) is onto. By Theorem 2 there exist bundles \( \mathcal{B}(T^2, R^3), \mathcal{B}_1(T^2, G_1) \) and injection \( j: \mathcal{B}_1 \rightarrow \mathcal{B} \) such that \( \chi \) is a characteristic homomorphism of this reduction. Theorem 4 then asserts that \( \mathcal{B}(T^2, R^3) \) does not have a cross section.

**References**


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