ON THE EXISTENCE OF CROSS SECTIONS IN
LOCALLY FLAT BUNDLES

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1. A principal bundle $\mathfrak{B}(X, G)$ with base space $X$ and group $G$ is called "locally flat" if the structural group $G$ can be reduced to a totally disconnected subgroup $G_1 \subset G$, or in other words: if there exists a totally disconnected subgroup $G_1 \subset G$, a principal bundle $\mathfrak{B}_1(X, G_1)$ and an injection $j: \mathfrak{B}_1 \rightarrow \mathfrak{B}$ [1]. For instance, if a differentiable bundle $\mathfrak{B}$ admits an infinitesimal connection whose curvature form is identically zero, then $\mathfrak{B}$ is locally flat [1, Reduction theorem, p. 37].

We are mainly interested in locally flat bundles with arcwise connected structural group $G$. The aim of this note is to give necessary and sufficient conditions for locally flat bundles to have a cross section.

2. We have to make use of certain more or less obvious properties of locally flat bundles which were discussed in a previous paper. As all bundles in the sequel will be principal bundles we shall omit the word "principal"; we assume all base spaces $X$ of bundles to be arcwise connected.

Definition. A reduction of a bundle $\mathfrak{B}(X, G)$ is the collection of a subgroup $G_1 \subset G$, bundle $\mathfrak{B}_1(X, G_1)$ and injection $j: \mathfrak{B}_1 \rightarrow \mathfrak{B}$. Suppose $G$ can be reduced to a totally disconnected subgroup $G_1 \subset G$. Corresponding to points of reference $x_0 \in X$ and $b_1 \in B_1$, where $B_1$ is the bundle space of $\mathfrak{B}_1$, there exists the characteristic homomorphism $\chi: \pi_1(X) \rightarrow G_1$ [2, p. 61]. We say that the reduction is "irreducible" if $\chi$ is onto.

Definition. Let $j: \mathfrak{B}_1(X, G_1) \rightarrow \mathfrak{B}(X, G)$ be a reduction, $G_1$ totally disconnected. $\chi: \pi_1(X) \rightarrow G_1$ is called a "characteristic homomorphism of the reduction," $H = \ker(\chi)$ the "kernel of the reduction."

3. From now on we assume $X$ to be arcwise connected, arcwise locally connected and semi locally 1-connected. This will make sure the existence of bundles over $X$ with totally disconnected group and prescribed characteristic homomorphism. In particular, to every invariant subgroup $N \subset \pi_1(X)$ belongs a regular covering $p_2: \tilde{X} \rightarrow X$ such that $\pi_1(\tilde{X}) \approx N$. If we put $P = \pi_1(X)/N$ then $\tilde{X}$ is the space of a bundle $\mathfrak{B}_2(X, P)$ with discrete group $P$, and the right action of $p \in P$ on $\tilde{x} \in \tilde{X}$—denoted by $\tilde{x} \cdot p$—is a "deckbewegung."

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Theorem 1. Let $\mathcal{B}(X, G)$ be locally flat, $\chi: \pi_1(X) \to G_1 \subseteq G$ a characteristic homomorphism and $H$ the kernel of a reduction $j: \mathcal{B}_1(X, G_1) \to \mathcal{B}(X, G)$. Let $N$ be an invariant subgroup of $\pi_1(X)$ such that $N \subseteq H$. $\chi$ induces a homomorphism $P \to G_1$ again denoted by $\chi$. Let $\tilde{X}$ be the covering space of $X$ corresponding to $N$. Then, $\mathcal{B}(X, G)$ has a cross section if and only if there exists a map $f: \tilde{X} \to G$ such that

\begin{equation}
(3.1) \quad f(\tilde{x} \cdot p) = f(\tilde{x})\chi(p) \quad \text{for all} \quad \tilde{x} \in \tilde{X} \quad \text{and} \quad p \in P.
\end{equation}

Theorem 2. Suppose a homomorphism $\chi: \pi_1(X) \to G_1 \subseteq G$ is given, and $G_1$ is totally disconnected. Then there exist bundles $\mathcal{B}(X, G)$, $\mathcal{B}_1(X, G)$ and injection $j: \mathcal{B}_1 \to \mathcal{B}$ such that $\chi$ is a characteristic homomorphism of this reduction. Furthermore, any two such bundles $\mathcal{B}(X, G)$ are equivalent.

For the proof of Theorems 1 and 2 we refer to [3].

4. Definition. Let $T^n$ denote the $n$-dimensional torus, $n \geq 0$; $T^0$ consists of one point. Let $G$ be a topological group and $G_1$ a subgroup $\subseteq G$. We say that $G_1$ is $n$-flat in $G$ if every bundle $\mathcal{B}(T^n, G)$ over $T^n$ whose structural group $G$ can be reduced to $G_1$, has a cross section.

Clearly, $G_1 = \{e\}$ is $n$-flat in $G$ for every $n$. If $G$ is arcwise connected, every subgroup of $G$ is 1-flat in $G$. If $G_1$ is $n$-flat in $G$, then $G_1$ is $p$-flat in $G$ for every $0 \leq p \leq n$. As will be shown later, $n$-flat subgroups which are totally disconnected are characterized by topological properties.

Theorem 3. Suppose $X$ is, in addition to being connected in the sense of §3, a normal space with the property that every covering of $X$ by open sets is reducible to a countable covering. Let $\mathcal{B}(X, G)$ be locally flat, $j: \mathcal{B}_1(X, G_1) \to \mathcal{B}(X, G)$ a reduction, and $H$ the kernel of the reduction. Suppose there exists an invariant subgroup $N \subseteq \pi_1(X)$ such that $N \subseteq H$ and $P = \pi_1(X)/N$ is a finitely generated free abelian group whose dimension is $n$. Then $\mathcal{B}$ has a cross section if $G_1$ is $n$-flat in $G$.

5. The proof of Theorem 3 will be preceded by two lemmas.

Lemma 1. Let $X$ and $N \subseteq \pi_1(X)$ be the same as in Theorem 3. Let $p_2: \tilde{X} \to X$ be the covering corresponding to $N$, $\{p_1, \ldots, p_n\}$ a basis of $P = \pi_1(X)/N, \{e_1, \ldots, e_n\}$ a basis of Euclidean space $E^n$. The correspondence $p_i \to e_i$ sets up an isomorphism $\psi: P \to Z^n$, where $Z^n$ is the vector group generated by $\{e_1, \ldots, e_n\}$. Then, there exists a map $t: \tilde{X} \to E^n$ such that $t(\tilde{x} \cdot p) = t(\tilde{x}) + \psi(p)$.

The composition of the natural homomorphism $\phi: \pi_1(X) \to P$ and $\psi$ is a homomorphism $\psi_\phi: \pi_1(X) \to Z^n \subseteq E^n, E^n$ being the additive group.
of $E^n$. By Theorem 2 there exist bundles $\mathcal{B}(X, E^n)$, $\mathcal{B}_1(X, Z^n)$ and injection $j: \mathcal{B}_1 \to \mathcal{B}$, such that $\psi_1$ is a characteristic homomorphism of this reduction. The topological space of $E^n$ is solid and $X$ is such that every bundle over $X$ with a solid fibre has a cross section [2, p. 55]. Hence, $\mathcal{B}(X, E^n)$ has a cross section and the map $t$ of the lemma is the map $g$ given by Theorem 1, the right hand side of (3.1) written additively and $\chi$ replaced by $\psi$.

**Lemma 2.** The situation being as in Theorem 3 and Lemma 1, let $T^n$ be the $n$-dimensional torus and $\{u_1, \ldots, u_n\}$ a basis of $\pi_1(T^n)$. $E^n$ is the universal covering space of $T^n$, and we can assume that the lifting of curves in the class $u_i$ provides curves with initial point $O \in E^n$ and endpoint $e_i \in E^n$. This induces an isomorphism $\pi_1(T^n) \approx Z^n$, with respect to which $\chi_1 = \chi^{-1}$ can also be interpreted as a homomorphism $\pi_1(T^n) \to G_1$. By Theorem 2 there exist bundles $\mathcal{B}(T^n, G)$, $\mathcal{B}_1(T^n, G_1)$ and injection $j_1: \mathcal{B}_1 \to \mathcal{B}$ such that $\chi_1$ is a characteristic homomorphism of this reduction.

Suppose now, $G_1$ is $n$-flat in $G$. Then the bundle $\mathcal{B}(T^n, G)$ has a cross section which, on the other hand, means that there exists a map $f_1: E^n \to G$ such that $f_1(v+z) = f_1(v)\chi_1(z)$ for all $v \in E^n$ and $z \in Z^n$. The composition of this $f_1$ with $t$ given by Lemma 1 is a map $f = f_1t: \tilde{X} \to G$ which has the property (3.1). Hence $\mathcal{B}(X, G)$ has a cross section and Theorem 3 is proved.

6. **Theorem 4.** Let the notations be as before. Assume the reduction in Theorem 3 to be irreducible. Suppose there exists a map $s: E^n \to \tilde{X}$ such that $s(v+z) = s(v) \cdot \chi^{-1}(z)$ for all $v \in E^n$ and $z \in Z^n$. Then $\mathcal{B}(X, G)$ has a cross section if and only if $G_1$ is $n$-flat in $G$.

The sufficiency is given by Theorem 3. Let $\mathcal{B}(X, G)$ have a cross section. There exists a map $f: \tilde{X} \to G$ with the property (3.1). The composition of $f$ with $s$ is a map $f_1 = fs: E^n \to G$ such that $f_1(v+z) = f_1(v)\chi_1(z)$ for all $v \in E^n$ and $z \in Z^n$. This means that $\mathcal{B}(T^n, G)$ has a cross section (Lemma 2 and Theorem 1). The proof of Theorem 4 will be complete after the correctness of the following has been shown.

**Lemma 3.** A totally disconnected subgroup $G_1 \subset G$ is $n$-flat in $G$ if there exist bundles $\mathcal{B}(T^n, G)$, $\mathcal{B}_1(T^n, G_1)$ and an irreducible reduction $j_1: \mathcal{B}_1 \to \mathcal{B}$ (we mean by this that all characteristic homomorphisms $\chi_1: \pi_1(T^n) \to G$ are onto) such that $\mathcal{B}(T^n, G)$ has a cross section.

The lemma will be proved later. We are going to explain now the notion of $n$-flatness in detail.

7. In the following, $G_1 \subset G$ will always be assumed to be totally
disconnected. Let \( \{ h_1, \cdots, h_n \} \) be a commutative set of \( n \) elements in \( G_1 \). We mean by this a set of \( n \) elements \( h_i \in G_1 \) such that \( h_j h_k = h_k h_j \). Let \( \{ e_1, \cdots, e_n \} \) be a basis of Euclidean space \( E^n \).

**Lemma 4.** If \( G_1 \) is \( n \)-flat in \( G(n \geq 1) \), then there exists a map \( f: E^n \rightarrow G \) such that

\[
(7.1) \quad f(v + e_i) = f(v) h_i \quad \text{for all } v \in E^n \text{ and } i = 1, \cdots, n.
\]

If \( \{ p_1, \cdots, p_n \} \) is a basis of \( \pi_1(T^n) \) we can assume that the lifting of curves in the class \( p_i \) provides curves from \( 0 \in E^n \) to \( e_i \in E^n \). The correspondence \( p_i \rightarrow h_i \) generates a homomorphism \( \chi_1: \pi_1(T^n) \rightarrow G_1 \). By Theorem 2 there exists a bundle \( \mathcal{B}(T^n, G) \) and a reduction of \( G \) to \( G_1 \) with characteristic homomorphism \( \chi_1 \). If \( G_1 \) is \( n \)-flat in \( G \), \( \mathcal{B}(T^n, G) \) has a cross section, and \( f \) in Lemma 4 is the map \( f \) given by Theorem 1.

Let now \( T^n(n \geq 0) \) be the \( n \)-dimensional torus and \( G \) a topological group. Let \( t_0 \in T^n \) be a point of reference and \( F^n \) the set of continuous mappings \( f: (T^n, t_0) \rightarrow (G, e) \). If \( f_1 \in F^n, f_2 \in F^n \), then \( f_1 \cdot f_2 \) denotes the mapping \( f_1 \cdot f_2(t) = f_1(t) \cdot f_2(t), \quad t \in T^n \), and \( f^{-1} \) the composition of \( f_1: T^n \rightarrow G \) with the map \( g \rightarrow g^{-1} \) in \( G \). Let \( \tau_n(G) \) denote the set of homotopy classes of mappings \( f \in F \). \( \tau_n(G) \) is a group, the product of two classes \( u_1 \) and \( u_2 \) being defined as the class of products \( f_1 \cdot f_2 \), with \( f_1 \in u_1, f_2 \in u_2 \). Clearly, \( \tau_1(G) = \pi_1(G) = \text{fundamental group of } G \).

Suppose then \( \{ h_1, \cdots, h_n \} \) is a commutative set of elements of \( G_1 \). If \( G_1 \) is \( n \)-flat in \( G \) and \( \{ e_1, \cdots, e_n \} \) is a basis of \( E^n \) there exists a map \( f: E^n \rightarrow G \) with property (7.1). We may even assume

\[
(7.2) \quad f(0) = e = \text{identity in } G,
\]

because, if \( f \) satisfies (7.1) and \( a = f(0) \) then \( g(v) = a^{-1} \cdot f(v) \) satisfies both (7.1) and (7.2). This being said, let \( h_{n+1} \in G_1 \) be such that \( h_i h_{n+1} = h_{n+1} h_i, \quad i = 1, \cdots, n \). Because of the commutativity, \( h_{n+1} \cdot f \cdot h_{n+1}: E^n \rightarrow G \) will again have properties (7.1) and (7.2). If two mappings \( f_1: E^n \rightarrow G \) and \( f_2: E^n \rightarrow G \) have properties (7.1) and (7.2), then \( f_1 \cdot f_2^{-1}: E^n \rightarrow G \) induces a map \( \tilde{g}: (T^n, t_0) \rightarrow (G, e) \) where \( t_0 = p_2(0) \) and \( p_2: E^n \rightarrow T^n \) is the covering, such that \( f_1 \cdot f_2^{-1} \cdot \tilde{g} = g \cdot p_2. \) Hence \( h_{n+1}^{-1} \cdot f_1 \cdot f_{n+1}^{-1} \cdot f^{-1} \) induces a map \( \tilde{f}_1: (T^n, t_0) \rightarrow (G, e) \) which lies in a homotopy class \( u \in \tau_n(G) \). If the elements \( h_1, \cdots, h_{n+1} \) are kept fixed, \( u \) still might depend on \( f \). Let \( f_1: E^n \rightarrow G \) be another map with properties (7.1) and (7.2), and \( \tilde{f}_1 \) be the map \( E^n \) induced by \( h_{n+1}^{-1} \cdot f_1 \cdot h_{n+1} \cdot f_1^{-1} \). Applying the reasoning above, \( f_1 \cdot f_1^{-1} = g \) induces a map \( \tilde{g} \in F^n \). From \( h_{n+1}^{-1} \cdot f_1 \cdot h_{n+1} \cdot f_1^{-1} = h_{n+1}^{-1} \cdot g \cdot h_{n+1} \cdot f_1 \cdot f_{n+1}^{-1} \cdot f^{-1} \cdot g \) follows \( \tilde{f}_1 = h_{n+1}^{-1} \cdot \tilde{g} \cdot h_{n+1} \cdot f_1 \cdot f_1^{-1} \cdot \tilde{g}^{-1} \). As \( h_{n+1}^{-1} \cdot \tilde{g} \cdot h_{n+1} \) is homotopic to \( \tilde{g} \) in \( F^n \) one gets \( u_1 = u_0 \cdot u_1 \cdot u_0^{-1} \), where \( u_1, u_0 \) is the homotopy class of \( \tilde{f}_1, \tilde{g} \) resp. In short, we have the following result.
Lemma 5. Let $G_i$ be $n$-flat in $G$, $n \geq 1$. The construction above assigns to every commutative set of $(n+1)$ elements $h_i \in G_i$, $i = 1, \ldots, (n+1)$, an equivalence class $\psi_n(h_1, \ldots, h_{n+1})$ under inner automorphisms of elements $\in \tau_n(G)$.

8. We adopt the convention that, if $\psi$ denotes an equivalence class under inner automorphisms of elements of a group $K$, $\psi = 0$ means it contains the identity $e \in K$.

Theorem 5. A totally disconnected group $G_1 \subset G$ is $(n+1)$-flat in $G(n \geq 1)$, if and only if it is $n$-flat in $G$ and $\psi_n(h_1, \ldots, h_{n+1}) = 0$ for every commutative set of elements in $G_i$. The statement: $G_i$ is $1$-flat in $G$, is equivalent to: $G_i$ lies in the arcwise connected component of $e \in G$.

The second part of Theorem 5 is obvious. As for the first part, let us assume that $G_i$ is $n$-flat in $G(n \geq 1)$ and $\psi_n(h_1, \ldots, h_{n+1}) = 0$. Let $\mathfrak{B}(T^{n+1}, G)$, $\mathfrak{B}_i(T^{n+1}, G_i)$ be bundles, $j: \mathfrak{B}_i \rightarrow \mathfrak{B}$ an injection and $\chi: \pi_1(T^{n+1}) \rightarrow G_i$ a characteristic homomorphism of this reduction. Let $p_2: E^{n+1} \rightarrow T^{n+1}$ be the universal covering, $\{p_1, \ldots, p_{n+1}\}$ a basis of $\pi_1(T^{n+1})$, $\{e_1, \ldots, e_{n+1}\}$ a basis of $E^{n+1}$ such that the lifting of closed curves in the class $p_i \in \pi_1(T^{n+1}, t_0)$ provides curves in $E^{n+1}$ from 0 to $e_i$, $t_0 = p_2(0)$. The set of elements $h_i = \chi(p_i) \in G_i$, $i = 1, \ldots, (n+1)$, is certainly commutative. As $G_i$ is $n$-flat in $G$ there exists by Lemma 4 a map $f_1: E^n \rightarrow G$ such that

$$f_1(v + e_i) = f_1(v)h_i,$$

for all $v \in E^n$ and $i = 1, \ldots, n$.

We may assume $f_1(0) = e \in G$. Denote by $g: E^n \rightarrow G$ the map $h_{n+1}^{-1} \cdot f_1 \cdot h_{n+1} \cdot f_1^{-1}$, by $\bar{g}: T^n \rightarrow G$ the induced map such that $g = \bar{g}p_2$, where $p_2: E^n \rightarrow T^n$ is the covering map. By the assumption $\psi_n(h_1, \ldots, h_{n+1}) = 0$ there exists a homotopy $h: (I \times T^n, t_0) \rightarrow (G, e)$ such that $h(1, t) = \bar{g}(t)$ and $h(0, t) = e$ for $t \in T^n$. Then $h$ defined by $h_1(v) = h_1(\rho, p_2(v))$ for $\rho \in I$ and $v \in E^n$, is a homotopy of $g$ such that

$$h(1, v) = g(v), h(0, v) = e = h_1(0, v), h_1(\rho, v + e_i) = h_1(\rho, v),$$

$i = 1, \ldots, n$.

Let now $w: I \rightarrow G$ be a curve connecting $e = w(0)$ and $w(1) = h_{n+1}$. Then $f_1$ defined by $f_1(\rho, v) = w(\rho)h_1(\rho, v)f_1(v)$ is a mapping $I \times E^n \rightarrow G$ which has the following properties

$$f_1(0, v) = f_1(v), f_1(\rho, v + e_i) = f_1(\rho, v)h_i \quad (i = 1, \ldots, n),$$

$$f_1(1, v) = f_1(0, v)h_{n+1}.$$  

Any vector $w \in E^{n+1}$ can be written as $\rho e_{n+1} + v$, $v \in E^n$. Let $m$ be an
integer and \( E_m \) the set \( \{ w = \rho e_{n+1} + v | m \leq \rho \leq (m + 1) \} \). We define mappings \( f_m: E_m \rightarrow G \) by \( f_m(w) = f^*(\rho - m, v)h^m_{n+1} \). Because of (8.2) one has \( f_m(w) = f_{m+1}(w) \) if \( w \in E_m \setminus E_{m+1} \), thus the collection \( \{ f_m \} \) determines one map \( f: E^{n+1} \rightarrow G \) with the property \( f(w + e_i) = f(w)h_i, \ i = 1, \cdots, (n + 1) \), where \( h_i = \chi(e_i) = \chi(e_i) \). But the existence of such a map asserts the existence of a cross section in \( \mathcal{B}(T^{n+1}, G) \) (Theorem 1). Hence \( G_i \) is \( (n + 1) \)-flat in \( G \).

Conversely suppose \( G_i \) to be \( (n + 1) \)-flat in \( G (n \geq 1) \). As was pointed out in connection with the definition of \( n \)-flatness, \( (n + 1) \)-flatness induces \( n \)-flatness. Let \( \{ h_1, \cdots, h_{n+1} \} \) be a commutative set of elements in \( G_i \). Using the same notations as before, the correspondence \( p_i \in \pi_1(T^{n+1}) \rightarrow h_i \) generates a homomorphism \( \chi: \pi_1(T^{n+1}) \rightarrow G_i \) into. By Theorem 2, \( \chi \) is a characteristic homomorphism of a certain reduction \( j: \mathcal{B}_1(T^{n+1}, G_i) \rightarrow \mathcal{B}(T^{n+1}, G) \) where \( \mathcal{B}(T^{n+1}, G) \) has a cross section because of the assumed \( (n + 1) \)-flatness of \( G_i \) in \( G \). By Theorem 1 there exists a map \( f: E_{n+1} \rightarrow G \) such that \( f(w + e_i) = f(w)h_i \) for \( w \in E^{n+1} \) and \( i = 1, \cdots, (n + 1) \). Denote by \( f_1 \) the restriction \( f | E^n \). We may assume \( f_1(0) = e \) (7.2). As \( w = \rho e_{n+1} + v, v \in E^n \), we can write \( f(w) = f_\rho(v) \). Denote by \( u(\rho) \) the expression \( f_\rho(0) = f(\rho e_{n+1}) \). Then \( h \) defined by \( h(\rho, v) = u^{-1}(\rho)f_\rho(v)f^{-1}(v) \) is continuous in \( E^{n+1} \) and represents for \( 0 \leq \rho \leq 1 \) a homotopy of \( h(1, v) = h_{n+1}^{-1}f_1(v)h_{n+1}f^{-1}(v) \) in \( h(0, v) = e \). As for all \( \rho u^{-1}(\rho)f_\rho(v + e_i)f^{-1}(v + e_i) = u^{-1}(\rho)f_\rho(v)f^{-1}(v), h \) will induce a homotopy \( h: (I \times T^n, t_0) \rightarrow (G, e) \) of the map \( g: (T^n, t_0) \rightarrow (G, e) \) induced by \( h_{n+1}^{-1} \cdot f_1 \cdot h_{n+1} \cdot f^{-1} \) in the map \( T^n \rightarrow e \). Therefore \( \psi_n(h_1, \cdots, h_{n+1}) = 0 \), and the proof of Theorem 5 is complete.

9. There still remains to prove Lemma 3. The situation is the following: two bundles \( \mathcal{B}(T^n, G), \mathcal{B}_1(T^n, G_i) \) together with a reduction \( j: \mathcal{B}_1 \rightarrow \mathcal{B} \) are given such that \( G_1 \subset G \) is totally disconnected, \( \mathcal{B}(T^n, G) \) has a cross section and, if \( \chi \) denotes a characteristic homomorphism, \( \chi: \pi_1(T^n) \rightarrow G_i \) onto. Using the same notations as before, there exists a map \( f: E^n \rightarrow G \) such that \( f(v + z) = f(v)\chi(z) \) for \( v \in E^n \) and \( z \in Z^n \) (Theorem 1) and \( f(0) = e \) (7.2). \( Z^n \) is the vector group generated by \( \{ e_1, \cdots, e_n \} \) and as such isomorphic to \( \pi_1(T^n) \). Any \( g_i \in G_i \) is the image under \( \chi \) of a certain \( z \in Z^n \). Let \( g_1 = \chi(z) \), then \( u(\rho) = f(\rho z) \) is a curve in \( G \) from \( e \) to \( g_1 \). This shows that \( G_1 \subset G \) arcwise connected component of \( G \) and hence is \( 1 \)-flat in \( G \) (Theorem 5). Suppose \( G_i \) is already proved to be \( p \)-flat in \( G \), \( 1 \leq p < n \). Let \( \{ h_{i_1}, \cdots, h_{p+1} \} \) be an ordered set of elements in \( G_i \); note that \( G_1 \) is abelian. Let \( v_i \in Z^n \) be such that \( \chi(v_i) = h_i \). If \( \{ e_1, \cdots, e_{p+1} \} \) is a basis of \( E^{p+1} \) then the correspondence \( e_i \rightarrow v_i \) generates a linear map \( \phi \) of \( E^{p+1} \) onto the vector-space spanned by \( \{ v_1, \cdots, v_{p+1} \} \). The composition \( f\phi: E^{p+1} \rightarrow G \) has
the property \( f(p + e_i) = f(p) h_i \) for \( w \in E^{p+1} \) and \( i = 1, \ldots, (p + 1) \). But now we have the same situation as in the second part of the proof of Theorem 5. The reasoning applied there leads to the conclusion \( \psi_p(h_1, \ldots, h_{p+1}) = 0 \). It follows from Theorem 5 that \( G_1 \) is \((p+1)\)-flat in \( G \).

10. Corollary 1. Every totally disconnected subgroup \( G_1 \) of an arcwise connected abelian group \( G \) is \( n \)-flat in \( G \) for every \( n \).

This follows directly from Theorem 5. One just has to observe that elements of \( \psi_n(h_1, \ldots, h_{n+1}) \) are homotopy classes of mappings of the form \( h_{n+1}^{-1} f h_n f^{-1} \).

Corollary 2. Let \( X \) be a topological space as in Theorem 3. If \( \pi_1(X) \) is a finitely generated free abelian group, then every locally flat bundle with base space \( X \) and arcwise connected abelian group \( G \) has a cross section.

This is a consequence of Theorem 3 and Corollary 1.

11. We wish to give an example of a locally flat bundle without cross section. Let \( R^3 \) denote the group of rotations in Euclidean space \( E^3 \) and \( \{ e_1, e_2, e_3 \} \) be an orthogonal basis in \( E^3 \). Define \( h_1 \) and \( h_2 \in R^3 \) by \( h_1(e_i) = -e_i, i = 1, 2, h_1(e_3) = e_3, h_2(e_1) = e_1, h_2(e_3) = -e_3 \). Clearly \( h_1 h_2 = h_2 h_1 \). The set of rotations in \( R^3 \) leaving \( e_3 \) fixed may be identified with \( R^2 \). If \( g \in R^2 \), then \( h_3^{-1} g h_2 = g^{-1} \), especially if \( f: I \rightarrow R^2 \) is a curve from \( e \) to \( h_1, h_3^{-1} f h_2 = f^{-1} \). As \( h_1^2 = e, u = h_3^{-1} f h_2 f^{-1} = f^{-2} \) is a closed curve in \( R^2 \), and we can choose \( f \) such that the homotopy class of \( u \) in \( R^2 \) generates \( \pi_1(R^2) \). But then \( u \) cannot be homotopic to zero in \( R^3 \) either, which means \( \psi_2(h_1, h_2) \neq 0 \). Hence the discrete group \( G_1 = \{ e, h_1, h_2, h_1 h_2 \} \) is not \( 2 \)-flat in \( R^3 \). If \( T^2 \) is now the two-dimensional torus we define \( \chi: \pi_1(T^2) \rightarrow G_1 \) by \( \chi(p_i) = h_i, i = 1, 2 \), where \( \{ p_1, p_2 \} \) in a basis of \( \pi_1(T^2) \). Note that \( \chi \) is onto. By Theorem 2 there exist bundles \( B(T^2, R^3), B_1(T^2, G_1) \) and injection \( j: B_1 \rightarrow B \) such that \( \chi \) is a characteristic homomorphism of this reduction. Theorem 4 then asserts that \( B(T^2, R^2) \) does not have a cross section.

References


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