A PROPERTY OF REGULAR MEASURES IN LOCALLY COMPACT HAUSDORFF SPACES[1]

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1. **Introduction.** Let G denote a locally compact Hausdorff topological space. $\mathcal{B}$ is the class of Borel sets of G, i.e., the σ-ring generated by the class $\mathcal{C}$ of compact subsets of G. Let $\mathfrak{D}$ be the class of sets whose intersection with every compact set is a Borel set. Note that $\mathfrak{D}$ is a σ-field containing the open sets of G. Given a measure μ (a non-negative countably additive set function not identically equal to zero) on $\mathcal{B}[\mathfrak{D}]$, denote by $\mathcal{B}_\mu[\mathfrak{D}_\mu]$ the class of sets which either belong to $\mathcal{B}[\mathfrak{D}]$ or which differ from a member of $\mathcal{B}[\mathfrak{D}]$ by a subset of a set of μ-measure zero.

A measure μ defined on $\mathcal{B}[\mathfrak{D}]$ is said to be regular if (i) $\mu(C)<\infty$ for every $C\in\mathcal{C}$ and (ii) for every $B\in\mathcal{B}[\mathfrak{D}]$, $\mu(B)=\sup\{\mu(C):C\in\mathcal{C},\ C\subseteq B\}$. Given a regular measure μ on $\mathcal{B}$ extend it to $\mathfrak{D}$ by defining $\mu(D)=\sup\{\mu(D\cap C):C \text{ compact}\}$ for all $D\in\mathfrak{D}$. Observe that μ so extended to $\mathfrak{D}$ is regular. Below we shall assume that either μ on $\mathfrak{D}$ is given or that μ on $\mathcal{B}$ is extended to $\mathfrak{D}$ as above.

The object of this paper is to prove the following

THEOREM. *To every regular measure μ on $\mathfrak{D}$ there corresponds a unique closed set A_μ, the carrier of μ, with $\mu(G\sim A_\mu)=0$ and $\mu(U)>0$ for every non-null relatively open subset U of A_μ.*

This theorem when G is compact is due to Wendel [1]. His proof is direct using heavily compactness of G.

2. PROOF OF THEOREM. As $\mu\not\equiv 0$ and as μ is regular we can find a compact set C with $\mu(C)>0$. As the space is locally compact and as C is a compact subset we can, by considering the covering of C by compact neighbourhoods of the points of C, find an open set $A\supseteq C$ with $\bar{A}$ compact. It is clear that $\mu(A)>0$ since $A\supseteq C$. A slight modification of the proof of [1, Lemma 3] then yields a compact set $A_1\subseteq\bar{A}$ such that every non-null relatively open subset of A_1 has positive μ-measure and $\mu(\bar{A}\sim A_1)=0$. Write $B=A\cap A_1$. As $A\in\mathcal{B}$ and $A_1\in\mathcal{B}$ it follows that $B\in\mathcal{B}$. Also $\mu(A\sim B)=0$ and every non-null relatively open subset of B has positive μ-measure.

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Consider now the collection of all open subsets A which admit of a subset $B \in \mathfrak{D}_\mu$ whose non-null relatively open subsets have positive measure and $\mu(A \sim B) = 0$. Let $\mathcal{E}$ denote the family of all such pairs (A, B). In view of earlier remarks this class is nonempty. Partially order this class by: $(A_1, B_1) \leqq (A_2, B_2)$ if and only if $A_1 \subseteq A_2$ and $B_1 = A_1 \cap B_2$.

We will show that every chain in $\mathcal{E}$ has an upper bound. It would then follow that $\mathcal{E}$ has a maximal element.

Let $\{(A_\alpha, B_\alpha), \alpha \in \Sigma\}$ be a chain from $\mathcal{E}$. Write $A = \bigcup_{\alpha \in \Sigma} A_\alpha$; $B = \bigcup_{\alpha \in \Sigma} B_\alpha$. A is thus an open set and hence belongs to $\mathfrak{D}$. We will show that $B \in \mathfrak{D}_\mu$ and that non-null relatively open subsets of B have positive measure.

Let μ^*, μ_* be respectively the outer and inner measures induced by μ and let K be any compact subset. Suppose that $C \subseteq A \cap K$, C a compact subset. Hence $C \subseteq A$ and $\{A_\alpha, \alpha \in \Sigma\}$ is an open covering for the compact set C. Therefore there exists a finite subcovering from this. As the A_α's are linearly ordered, it follows that $C \subseteq A_\theta$ for some $\theta \in \Sigma$. Now, $\mu(C) = \mu(C \cap A_\theta) = \mu(C \cap B_\theta)$ since $\mu(A_\theta \sim B_\theta) = 0$. Write $C_1 = C \cap B_\theta$. Hence C_1 is a Borel set, $\mu(C) = \mu(C_1)$ and $C_1 \subseteq B \cap K$. Regularity of μ now gives $\mu_*(B \cap K) \geqq \mu(A \cap K)$. That $\mu^*(B \cap K) \leqq \mu(A \cap K)$ follows from the fact that $B \cap K \subseteq A \cap K$. Hence

$$\mu^*(B \cap K) = \mu_*(B \cap K) = \mu(A \cap K).$$

This implies that $B \cap K \in \mathfrak{B}_\mu$. As the compact set K is arbitrary, $B \in \mathfrak{D}_\mu$. Also $\mu(B \cap K) = \mu(A \cap K)$. As μ is regular, we have $\mu(A \sim B) = 0$.

Let V be any open set with $B \cap V$ non-null. Hence there is a B_α, $\alpha \in \Sigma$ such that $B_\alpha \cap V$ is non-null. Therefore

$$\mu(B \cap V) \geqq \mu(B_\alpha \cap V) > 0.$$

Thus $(A, B) \in \mathcal{E}$. That $(A_\alpha, B_\alpha) \leqq (A, B)$ for all $\alpha \in \Sigma$ is evident. We have therefore proved that every chain from $\mathcal{E}$ has an upper bound. This implies that $\mathcal{E}$ has a maximal element. Let this be (M, N).

We claim $M = G$. For, if not, two cases can arise.

Case (i) $\mu(G \sim M) = 0$. In this case we see immediately that $(G, N) \in \mathcal{E}$. Further $(M, N) \leqq (G, N)$ and the two elements are not the same, thus contradicting the maximality of (M, N).

Case (ii) $\mu(G \sim M) > 0$. By the regularity of μ, there exists therefore a compact set $K \subseteq (G \sim M)$ with $\mu(K) > 0$. As explained at the beginning of the proof, we can find an open set $U \supseteq K$ such that $\overline{U}$ is compact. From all this we conclude that there exists a point $x \in (G \sim M)$

and an open set U such that $x \in U$, $\overline{U}$ is compact and $\mu(U) > 0$. We can then find a Borel set V such that $(U, V) \in \mathcal{E}$. Let $M_1 = M \cup U$ and $N_1 = N \cup V_1$ where $V_1 = (V \sim M)$. From the fact that non-null relatively open subsets of N and of V have positive measures we see that N_1 has this property too. Notice also that M_1 is open, $N_1 \subseteq M_1$ and $\mu(M_1 \sim N_1) = 0$. Thus $(M_1, N_1) \in \mathcal{E}$. Obviously however $M \subset M_1$ and $N = M \cap N_1$. So $(M, N) \leqq (M_1, N_1)$ and these two elements of $\mathcal{E}$ are not the same. This contradicts the maximality of (M, N). Therefore $G = M$, as we claimed.

Let $A_\mu = \overline{N}$. Then $\mu(G \sim A_\mu) = 0$.

Every open set having a non-null intersection with $\overline{N}$ has a non-null intersection with N. Therefore every non-null relatively open subset of A_μ has positive measure.

To prove the uniqueness of A_μ, assume, if possible, a second closed set $A \neq A_\mu$ such that $\mu(G \sim A) = 0$ and such that every non-null relatively open subset of A has positive μ-measure. If $A \not\subseteq A_\mu$, then $A \sim A_\mu$ is a non-null relatively open subset of A. Therefore $0 < \mu(A \sim A_\mu) \leqq \mu(G \sim A_\mu) = 0$, a contradiction. Hence $A \subseteq A_\mu$. Similarly $A_\mu \subseteq A$. Thus $A = A_\mu$ and the proof is complete.

Reference

1. J. G. Wendel, *Haar measure and the semi-group of measures on a compact group*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 923–929.

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