A NOTE ON MATRIX RICCATI SYSTEMS

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1. Let $A(t)$ be an $n \times n$ matrix, continuous on an interval $I$; let $n_1, \ldots, n_k$ be positive integers such that $\sum_{i=1}^k n_i = n$. Let $A$ be partitioned into submatrices $A_{ij}$ which are $n_i \times n_j$, $(i, j = 1, \ldots, k)$; let $E_m$ be the identity matrix of order $m$; and let $A_m = (A_{m1} \cdots A_{mk})$. In this note we consider the matrix Riccati system with side condition

$$Y' = -YA_mY + AY, \quad Y_m(t_0) = E_m,$$

where $Y = \text{col}(Y_1, \ldots, Y_k)$ and $Y_i$ is $n_i \times n_m$. This equation is derived in a natural way as a generalization of the so-called Riccati system [1; 2]. Mainly we generalize some results of Levin [3], who treats the equation

$$\Gamma' = -\Gamma G_3 \Gamma - \Gamma G_4 + G_1 \Gamma + G_2,$$

where $G_1, G_2, G_3,$ and $G_4$ are $n_1 \times n_1, n_1 \times n_2, n_2 \times n_1,$ and $n_2 \times n_2$ respectively, and $\Gamma$ is $n_1 \times n_2$. To see that (2) is a case of (1), take $m = 2$ and

$$A = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}, \quad Y(t) = \begin{pmatrix} \Gamma(t) \\ E_{n_2} \end{pmatrix}.$$

For other related results see [4].

2. Let $X$ be $n \times n$ and such that

$$X' = AX.$$

The partitioning of $A$ induces a partitioning of $X$ into submatrices $X_{ij}$ $(i, j = 1, \ldots, k)$. Let $X_m = \text{col}(X_{1m} \cdots X_{km})$. Then

$$X_m' = AX_m,$$

and, at least formally,

$$(X_m^{-1})' = -X_m^{-1}A_mX_m^{-1}.$$

Thus, if $X_m^{-1}$ exists on some interval $I_0 \subset I$, $X_mX_m^{-1}$ is a solution on $I_0$ of (1). Further, if $Y$ is a solution of the differential equation of (1), $Y_m$ satisfies an equation of the form $P' + HP = H$; thus from classical

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existence and uniqueness theorems (1) has a local existence and uniqueness theorem and \( Y_n(t) = E_{n_m} \). Gathering together these remarks, we have

**Theorem 1.** Let \( X(t) \) be the solution of (3) such that \( X(t_0) = E_n \). Then the general solution near \( t_0 \) of (1) is \( X(t)C(\sum_{i=1}^m X_{mi}C_i)^{-1} \), where \( C \) is constant and arbitrary, \( C = \text{col}(C_1 \cdots C_k) \), \( C_i \) is \( n_i \times n_m \), and \( C_m \) is nonsingular.

3. Now let \( n = rk \) and \( n_i = r \) (\( i = 1, \cdots, k \)). We proceed to write the general solution of (1) in terms of solutions of (1) rather than solutions of (3). Let \( U_1, \cdots, U_k \) denote solutions of (1); let \( U = (U_1 \cdots U_k) \); let \( Z = \text{diag}(Z_j) \), where \( Z_j \) is \( r \times r \) and

\[
Z_j = A_m U_j Z_j \quad (j = 1, \cdots, k).
\]

**Theorem 2.** Let \( U(t_0) \) and \( Z(t_0) \) be nonsingular; let \( C = \text{col}(C_j) \), the \( C_j \) being \( r \times r \), constant, and arbitrary except that \( \sum_{j=1}^r Z_j(t_0)C_j \) is nonsingular. Then the general solution of (1) can be written near \( t_0 \) as

\[
Y = UZC \left( \sum_{j=1}^k U_m Z_j C_j \right)^{-1}.
\]

**Proof.** It is easily verified that \( UZ \) satisfies (3) and that, if \( U(t_0) \) is nonsingular, \( U(t) \) is nonsingular where it exists; the conclusion then follows.

Now let \( U_{hij} = U_{hi} - U_{kj} \) and \( V_{hij} = U_{hij}^{-1} \). It is easily verified that \( (U_i - U_j)V_{hij} \) satisfies an equation of the form (1). Thus, if

\[
[(U_i - U_j)V_{hij} - (U_i - U_j)V_{hij}]_{t=t_0} = 0,
\]

it is identically zero near \( t_0 \). Hence, if (8) is satisfied, \( V_{hij} U_{hij} \) satisfies

\[
W' = A_m U_j W - W A_m U_q.
\]

Now fix \( q \); we may as well take \( q = 1 \).

**Theorem 3.** Let \( |U(t_0)| \neq 0 \). Let \( i = k + j - 1 \) (\( j = 2, \cdots, k \)). Let \( U_i \) be a solution of (1) such that, for some \( r_j \) (\( 1 \leq r_j \leq k \), \( r_j \neq m \)), \( U_{rij} \) and \( U_{rijq} \) are nonsingular at \( t_0 \) and (8) is satisfied for \( q = 1 \) and \( h = r_j \). Let \( |Z_i(t_0)| \neq 0 \). Then the general solution of (1) can be written in the form (7), where \( Z_j = V_{rij} U_{rij} Z_1 \) (\( j = 2, \cdots, k \)).

**Proof.** Note that \( Z_i = A_m U_i Z_1 \) and that \( V_{rij} U_{rij} \) satisfies (9) with \( q = 1 \); then apply Theorem 2.

We remark that, given \( q, j, \) and \( h \neq k \), there does exist a set of initial values for the \( U_i \) which satisfies (8). However, if for \( U_j \) and
$U_p (p \neq j)$ the same initial values for $U_i$ satisfy (8), there is a linear combination of columns of $U_i$, $U_j$, and $U_p$ which is zero. Thus, since $|U(t_0)| \neq 0$, each $U_j$ requires a distinct $U_i$. Thus, although we need not eliminate all of the $Z_j$'s ($j = 2, \cdots, k$) in the general solution, we do need a distinct $U_i$ for each one we do eliminate.

For $k = 2$ we have Theorem 5 of [3]. For $r = 1$ and general $k$ we have a form for $Y$ different from that given in [2].

BIBLIOGRAPHY


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