ON INFINITE PROLONGATIONS OF DIFFERENTIAL SYSTEMS

H. H. JOHNSON

From any exterior differential system with independent variables, it will be shown how to construct, on an infinite dimensional space, an equivalent completely integrable differential system. Local solutions are discussed in the real or complex analytic case.

The calculus of exterior differential systems will be used freely [1; 2]. Prolongations are clearly described in [3]. All functions and forms are assumed infinitely differentiable unless otherwise restricted.

1. Infinite prolongations. Let $E^{n+m}$ be $n+m$-dimensional real/complex Euclidean space. Let $\Sigma$ be any real/complex infinitely differentiable closed exterior differential system with dependent variables $z^1, \ldots, z^m$ and independent variables $x^1, \ldots, x^n$, defined on an open subset $D$ of $E^{n+m}$.

The first prolongation, $\Sigma'$, of $\Sigma$, can be accomplished as follows. Everywhere in $\Sigma$, replace $\Phi^\alpha$ by $\sum_{j=1}^n z_j^i dx^j$, $\lambda = 1, \ldots, m$, where $z_j^i$ are $nm$ new variables. Using the exterior calculus, the coefficients of the independent terms in the resulting forms, together with the 0-forms of $\Sigma$, and the 1-forms $dz_j^i$, together generate $\Sigma'$.

Thus, $\Sigma'$ consists of

1. a collection of 0-forms or functions, $\{\Phi_1^\alpha(x^i, z^i, z_j^i) ; \alpha = 1, \ldots, \alpha_1\}$ and their exterior derivatives, $\{d\Phi_1^\alpha ; \alpha = 1, \ldots, \alpha_1\}$,

2. the 1-forms $dz_j^i - z_k^i dx^j$ and their derivatives, $dz_j^i - z_k^i dx^j$.

The second prolongation, $\Sigma''$, is obtained from $\Sigma'$ by replacing $dz_j^i$ with $z_k^i dx^j$, where the $z_k^i = z_k^i$ are $mn(n-1)/2$ new variables. $\Sigma''$ consists of

1. the 0-forms of $\Sigma'$, $\{\Phi_2^\alpha ; \alpha = 1, \ldots, \alpha_1\}$, and new functions $\{\Phi_2^\alpha ; \alpha = 1, \ldots, \alpha_2\}$, together with the derivatives $\{d\Phi_2^\alpha ; \alpha = 1, \ldots, \alpha_2\}$,

2. $dz_j^i - z_k^i dx^j, dz_j^i - z_k^i dx^j, dz_j^i - z_k^i dx^j$. It is important that $d\Phi_2^\alpha \equiv 0$ modulo $\Sigma''$, $\alpha = 1, \ldots, \alpha_1$, and $d(dz_j^i - z_k^i dx^j) \equiv 0$ modulo $\Sigma''$. The former is a consequence of the prolongation process, the latter of the symmetry $z_{jk} = z_{kj}$.

Received by the editors August 2, 1960.
1 Research sponsored by OOR Contract DA-36-ORD-2164.

588
ON INFINITE PROLONGATIONS OF DIFFERENTIAL SYSTEMS

2(r), the rth prolongation, will be a closed system in x' = (x', z', \ldots, z'^r), where q = 1, 2, \ldots, r, and each z'^r is symmetric in the lower indices. \( \Sigma^{(r)} \) consists of

1. 0-forms \( \{ \Phi^r_1; \alpha = 1, \ldots, \alpha_1 \}, \{ \Phi^r_2; \alpha = 1, \ldots, \alpha_2 \}, \ldots, \{ \Phi^r_a; \alpha = 1, \ldots, \alpha_a \} \), and 1-forms \( \{ d\Phi^r_1; \alpha = 1, \ldots, \alpha_r \} \).

2. \( dx^1 - z^1 dx^i, dx^2 - z^2 dx^i, \ldots, d\sum_{i=1}^{q-1} z^i dx^i, \ldots, dx^q - z^q dx^i \).

Continued indefinitely, the result is an infinite sequence of 0- and 1-forms which may be written, grouping the 0-forms together, as a system \( \Sigma^* \) of

1. 0-forms \( \psi^\beta(x^1, z^1, \ldots, z^q, \ldots) \), \( \beta = 1, 2, \ldots, \) each depending on an only finite number of variables,

2. \( dz^1 - z^1 dx^i, \ldots, dz^q - z^q dx^i, \ldots, \) The \( z^q \) are symmetric in lower indices and may assume any real/complex values, while the \( (x^i, z^i) \in D \). Each \( \psi^\beta \) is infinitely differentiable. Most important, \( \Sigma^* \) is completely integrable, i.e., for any 0- or 1-form \( \omega \in \Sigma^* \), there is a finite subset, \( \Sigma^*_o \) of \( \Sigma^* \), such that

\[ d\omega = 0 \text{ modulo } \Sigma^*_o. \]

A solution of \( \Sigma^* \) consists of infinitely differentiable functions \( z^1(x^1), \ldots, z^q(x^1), \ldots, \) defined on a neighborhood \( N \) in \( \mathbb{E}^n \), so that \( (x^1, z^1(x^1)) \in D \) when \( (x^1) \in N \), and which causes every form in \( \Sigma^* \) to vanish identically when \( z^1, dz^1, \ldots \) are replaced by \( z^1(x^1), dz^1(x^1), \ldots \) respectively. Vanishing of the forms in (2) in \( \Sigma^* \) is equivalent to

\[ \frac{\partial z^\lambda(x^1)}{\partial x^1} \cdots \frac{\partial z^\lambda(x^1)}{\partial x^q} = z^\lambda_{ij_1 \ldots ij_q}(x^1). \]

There is a one-to-one correspondence between the solutions of \( \Sigma \) and \( \Sigma^{(r)} \) [3], and it is easy to see that this property extends to \( \Sigma^* \), the correspondence being given by (A).

2. Analytic case. Assume that the forms in \( \Sigma \) are real/complex analytic, i.e., all functions appearing as coefficients have this property. Then all forms in \( \Sigma^* \) are analytic. Let \( (j_1, \ldots, j_q) \) be a sequence of \( q \) integers, \( 1 \leq j_k \leq n \). If, among these, there are exactly \( \rho_1 \) ones, \( \rho_2 \) twos, \ldots, and \( \rho_n \) n's, let \( \rho(j_1, \ldots, j_q) = \rho_1! \rho_2! \cdots \rho_n! \).

A sequence \( (u^i, v^1, \ldots, v^1_{ij_q}, \ldots) \) of real/complex numbers, where the \( v^1 \) are symmetric in the lower indices, is called a zero of \( \Sigma^* \) if, for all \( \beta, \psi^\beta(u^i, v^1, \ldots, v^1_{ij_q}, \ldots) = 0 \).

Theorem. Let \( (u^i, v^1, \ldots, v^1_{ij_q}, \ldots) \) be a zero of \( \Sigma^* \), where
is a bounded sequence for some positive real numbers \( R_1, \ldots, R_n \), and 
\( \rho(j_1, \ldots, j_q) = \rho_1! \cdots \rho_q! \). Then there exists a unique solution,

\[
z^\lambda(x^i, \ldots, z_{j_1} \ldots j_q(x^i), \ldots)
\]

of \( \Sigma^\ast \) defined in a neighborhood of \( (u^i) \) and satisfying the initial conditions

\[
z^\lambda(u^i) = u^i, \ldots, z_{j_1} \ldots j_q(u^i) = v_{j_1} \cdots j_q.
\]

**Proof.** Let

\[
z^\lambda(x^i) = \sum_{\rho_1=0}^{\infty} \cdots \sum_{\rho_n=0}^{\infty} a_{\rho_1, \ldots, \rho_n}(x^i - u^i)^{\rho_1} \cdots (x^n - u^n)^{\rho_n}
\]

(B)

\[
z_{j_1} \ldots j_q(x^i) = \frac{\partial^q z^\lambda(x^i)}{\partial x^{i_1} \cdots \partial x^{i_q}}.
\]

These functions converge for \( |x^i - u^i| < R_j \), satisfy the initial conditions, and cause the 1-forms \( dz^\lambda - z'_{j} \, dx^i, \ldots, dz^\lambda_{j_1} \ldots j_q - z^\lambda_{j_1} \ldots j_q \, dx^i, \ldots \) to vanish.

Let \( \tilde{\psi}^\beta(x^i) \) denote \( \psi^\beta \) after \( z^\lambda, \ldots, z_{j_1} \ldots j_q, \ldots \) are replaced by the corresponding functions (B). By hypothesis, \( \tilde{\psi}^\beta(u^i) = 0 \). Since \( \Sigma^\ast \) is completely integrable,

\[
d\psi^\beta \equiv 0 \text{ modulo } \Sigma^\ast
\]

hence,

\[
\frac{\partial \psi^\beta}{\partial x^i} + \frac{\partial \psi^\beta}{\partial z^\lambda} z_j + \cdots + \frac{\partial \psi^\beta}{\partial z_{j_1} \ldots j_q} z_{j_1} \ldots j_q + \cdots \equiv 0 \text{ modulo } \{ \psi^\beta \},
\]

and the sums on both sides are finite. Hence, there are analytic functions \( A_j^\beta(x^i) \) such that

\[
\frac{\partial \psi^\beta}{\partial x^i} = A_j^\beta \psi^\gamma.
\]
Therefore, \((\partial \Psi^\beta / \partial x^i)(u^i) = 0\), all \(\beta\). In the same way

\[
\frac{\partial^2 \Psi^\beta}{\partial x^i \partial x^k} = A_{j\gamma}^\beta \frac{\partial \Psi^\gamma}{\partial x^k} + A_{j\gamma}^\beta \frac{\partial}{\partial x^k} \Psi^\gamma,
\]

implies \((\partial^2 \Psi^\beta / \partial x^i \partial x^k)(u^i) = 0\), etc. Thus, the Taylor’s expansion of \(\Psi^\beta\) at \((u^i)\) is zero. The initial conditions thus determine uniquely the \(x^b(x^i)\), hence the solution. Q.E.D.

**Bibliography**


**Princeton University**