MATHEMATICAL PEARLS

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FREE PRODUCTS WITH AMALGAMATION AND 3-MANIFOLDS

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Throughout this paper M will be a compact 3-manifold, possibly with boundary.

If M is a closed 3-manifold and $\pi_2(M)=0$, then $\pi_1(M)$ is not a nontrivial free product. This widely known result can be deduced from the theory of ends [1, p. 100; **2**, Satz VI]. A generalisation of the result is proved here, without using ends.

LEMMA. *If $\pi_1(M)$ is infinite and $\pi_2(M)=0$, then M is aspherical.*

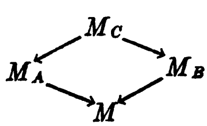
This can be seen by using the Hurewicz isomorphism theorem in the universal cover of M.

THEOREM. *Let $\pi_1(M)=A*_C B$ ($C\neq A, B$), i.e. a nontrivial free product with amalgamation. Let $\pi_2(M)=0$. Then, for any simple coefficient group Λ,*

$$H_3(A;\Lambda) = H_3(B;\Lambda) = H_3(C;\Lambda) = 0$$

and $H_2(C;\Lambda)$ contains a subgroup isomorphic to $H_3(M;\Lambda)$.

Since $\pi_1(M)$ is infinite, we know from the lemma that M is aspherical. Therefore, in the following diagram of covering spaces, all spaces are aspherical.



where $\pi_1(M_G)\approx G$, $G=A$, B or C. The groups A, B and C each have infinite index in $A*_C B$. Therefore M_A, M_B and M_C are infinite sheeted coverings of M. So $H_3(G;\Lambda)\approx H_3(M_G;\Lambda)=0$ where $G=A$, B or C.

We construct the mapping cylinders of $M_A\leftarrow M_C$ and $M_C\rightarrow M_B$ and identify the two subspaces M_C. Call the resulting space M'. M' is aspherical, as can be seen using the Hurewicz isomorphism theorem

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and the Mayer-Vietoris sequence in the universal cover of M'. Also $\pi_1(M') \approx A *_C B$ by van Kampen's theorem. Therefore M' is homotopy equivalent to M. From the Mayer-Vietoris sequence for M', M_A, M_B and M_C, we have the exact sequence

$$0 \rightarrow H_3(M'; \Lambda) \rightarrow H_2(M_C; \Lambda).$$

The theorem follows.

COROLLARY. *If M is closed and $C=1$, then $\pi_2(M) \neq 0$.*

For $H_2(C; Z_2)=0$ and $H_3(M; Z_2)=Z_2$.

REFERENCES

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2. E. Specker, *Die erste Cohomologiegruppe von Überlagerungen und Homotopie-Eigenschaften dreidimensionaler Mannigfaltigkeiten*, Comment. Math. Helv. vol. 23 (1949) pp. 303–333.

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