ON COMPONENTS IN SOME FAMILIES OF SETS

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1. Introduction. Helly's well-known theorem [3] states that all the members of a family \( \mathcal{C} \) of compact convex subsets of the Euclidean \( n \)-space \( E^n \) have a point in common provided every \( n + 1 \) members of \( \mathcal{C} \) have a common point. On the other hand (Motzkin, cf. Hadwiger-Debrunner [2] for further reference), there exists no (finite) number \( h \) with the following property: If \( \mathcal{K} \) is a family of subsets of \( E^n \) (even of \( E^1 \)) such that each member of \( \mathcal{K} \) is the union of at most two disjoint, compact, convex sets, and such that every \( h \) members of \( \mathcal{K} \) have a common point, then all the members of \( \mathcal{K} \) have a common point.

A consideration of the examples which establish the nonexistence of \( h \) led to the idea that there might exist theorems of Helly's type for such families \( \mathcal{K} \) if an additional condition is imposed on \( \mathcal{K} \): the intersection of any two members of \( \mathcal{K} \) should also be representable as the union of at most two disjoint, compact, convex sets. The present paper contains a theorem in this direction together with related results on families \( \mathcal{K} \) whose elements are disjoint unions of members of another family \( \mathcal{C} \).

In §2 we give the definitions of the properties we consider, and the statements of our main results. The proofs follow in §3. Remarks, examples, and counter-examples are given in §4.

2. Definitions and results. We shall deal mainly with families of subsets of some set, on whose nature nothing is assumed.

For a set \( A \) or an ordinal \( \mu \) we denote by \( \text{card} \ A \) resp. \( \text{card} \ \mu \) the corresponding cardinal. Thus, for a family of sets \( \mathcal{C} = \{ C_\alpha : \alpha \in A \} \) we have \( \text{card} \ \mathcal{C} = \text{card} \ A \). The letter \( \omega \) is used only for initial ordinals.

For a family of sets \( \mathcal{C} = \{ C_\alpha : \alpha \in A \} \) we put \( \pi \mathcal{C} = \bigcap_{\alpha \in A} C_\alpha \) and \( \sigma \mathcal{C} = \bigcup_{\alpha \in A} C_\alpha \).

We define \( K = C_1 + C_2 \) to be an abbreviation for the statement "\( K = C_1 \cup C_2 \) and \( C_1 \cap C_2 = \emptyset \)." Similarly, for \( \mathcal{C} = \{ C_\alpha : \alpha \in A \} \), we write \( K = \sum_{\alpha \in A} C_\alpha = \Sigma \mathcal{C} \) for "\( K = \sigma \mathcal{C} \) and \( C_\alpha \cap C_\beta = \emptyset \) for all \( \alpha, \beta \in A \) with \( \alpha \neq \beta \)."

If \( K = \Sigma \mathcal{C} \), each member of \( \mathcal{C} \) is a component of \( K \) and \( \Sigma \mathcal{C} \) is a decomposition of \( K \).

Presented to the Society, November 25, 1960; received by the editors September 12, 1960.

1 The preparation of this paper was sponsored in part by the National Science Foundation, and by the Office of Naval Research.

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For any family $\mathcal{C}$ and any cardinal $\gamma$ let $[\mathcal{C}]_\gamma = \{ \mathcal{C}' : \mathcal{C}' \subseteq \mathcal{C}, \text{card } \mathcal{C}' < \gamma + 1 \}$ and $[\mathcal{C}] = \{ \mathcal{C}' : \mathcal{C}' \subseteq \mathcal{C} \}$. For $K \in [\mathcal{C}]$ let $c(K) = \min \{ \text{card } \mathcal{C}' : K = \sum \mathcal{C}', \mathcal{C}' \subseteq \mathcal{C} \}$.

This paper deals with some properties of families of sets which we proceed to define.

**Definition 1.** A family $\mathcal{C}$ is $\gamma$-intersectional (for a finite or infinite cardinal $\gamma \geq 1$) if for every subfamily $\mathcal{C}' \subseteq \mathcal{C}$ with $\text{card } \mathcal{C}' < \gamma + 1$ we have $\mathcal{C}' \subseteq \mathcal{C}$. The family $\mathcal{C}$ is intersectional if it is $\gamma$-intersectional for every $\gamma \geq 1$.

Obviously, if $\gamma^* \leq \gamma$ and $\mathcal{C}$ is $\gamma$-intersectional, it is $\gamma^*$-intersectional as well. Every family is 1-intersectional; every 2-intersectional family is $\aleph_0$-intersectional.

**Definition 2.** A family $\mathcal{C}$ is $\gamma$-nonadditive (for a finite or infinite cardinal $\gamma \geq 2$) if for every subfamily $\mathcal{C}' \subseteq \mathcal{C}$, with $0 < \mathcal{C}'$ and $1 < \text{card } \mathcal{C}' < \gamma + 1$, such that $\sum \mathcal{C}'$ is defined, we have $\sum \mathcal{C}' \neq \mathcal{C}$. The family $\mathcal{C}$ is nonadditive if it is $\gamma$-nonadditive for every $\gamma \geq 2$.

**Examples.** The family of all closed [open] subsets of $\mathbb{E}^n$ is intersectional [$\aleph_0$-intersectional]. The family of all connected and open [compact] subsets of $\mathbb{E}^n$ is nonadditive [$\aleph_0$-nonadditive; see [4]]. In the set of ordinals $\{ \alpha : \alpha < \omega, \text{card } \omega = k \}$, for any $k > \aleph_0$, all segments of the form $[\alpha, \beta]$ or $[\beta, \omega)$, where $\alpha, \beta$ are limit-ordinals, form a family $\mathcal{S}$ which is intersectional and nonadditive. For any set $S$ with $\text{card } S = k \geq \aleph_0$ the family of all subsets of $S$ with complements of cardinal less than $k$ is $\aleph_0$-intersectional and nonadditive.

**Definition 3.** A family $\mathcal{C}$ has the Helly property of order $h$ with limit $\gamma$ (\(h, \gamma\) cardinals with $2 \leq h < \gamma$) if for each subfamily $\mathcal{C}' \subseteq \mathcal{C}$, with $\text{card } \mathcal{C}' < \gamma + 1$, the condition $"\pi \mathcal{C}' \neq \emptyset"$ for all $\mathcal{C}' \subseteq \mathcal{C}'$, with $\text{card } \mathcal{C}' < h + 1$ implies $\pi \mathcal{C}' \neq \emptyset$. The family $\mathcal{C}$ has the unlimited Helly property of order $h$ if it has the Helly property of order $h$ with limit $\gamma$ for every $\gamma > h$.

**Examples.** The family of all compact subsets of any topological space has the unlimited Helly property of order $\aleph_0$. The family of convex subsets of $\mathbb{E}^n$ has the Helly property of order $n + 1$ with limit $\aleph_0$; that of compact convex subsets has the unlimited Helly property of order $n + 1$ (Helly’s theorem). The family of all closed segments $[\alpha, \beta]$ of a well-ordered set has the unlimited Helly property of order 2; if segments $[\alpha, \mu)$, for a limit ordinal $\mu$, are included, the family has the Helly property of order 2 with limit $\text{card } \mu$.

The first theorem gives a criterion for the uniqueness of the decomposition of $K$.

**Theorem 1.** Let $\mathcal{C} = \{ C_\alpha : \alpha \in A \}$ be 2-intersectional and $\gamma$-nonadditive, and $K \in [\mathcal{C}]$. If $K = \sum_{\alpha' \in A'} C_{\alpha'}$ with $A' \subseteq A$, card $A' < \gamma + 1$,
and $C_a \neq \emptyset$ for all $a' \in A'$, and if $K = \sum_{a' \in A'} C_a'$ with $A'' \subset A$, card $A'' < \gamma + 1$, and $C_a'' \neq \emptyset$ for all $a'' \in A''$, then there exists a one-to-one map $\phi$ from $A'$ onto $A''$ such that $C_a = C_{\phi(a')}$ for all $a' \in A'$. In other words, the components of $K$ are uniquely determined.

As an immediate corollary we have:

**Corollary.** Let $\mathcal{C}$ be 2-intersectional and $\gamma$-nonadditive, and let $K \in [\mathcal{C}]_\gamma$ (i.e., $c(K) \leq n$), where $n$ is a finite cardinal and $\gamma \geq n$. Let $K^* \in [\mathcal{C}]_\gamma$, $K \subset K^*$, and let some $n$ different components of $K^*$ each have a nonempty intersection with $K$. Then different components of $K$ are contained in different components of $K^*$, and, in particular, $c(K) = n$.

Obvious examples show that the corollary may fail for infinite $n$.

The next theorem shows that $[\mathcal{C}]_7$ is, in a sense, weakly intersectional: if the intersections of all members of certain subfamilies of $\mathcal{X} \subset [\mathcal{C}]_7$, belong to $[\mathcal{C}]_7$, then for each subfamily of $\mathcal{X}$ the intersection of its members belongs to $[\mathcal{C}]_7$.

**Theorem 2.** Let $\mathcal{C}$ be $\gamma$-intersectional and $\gamma'$-nonadditive, $\mathcal{X} \subset [\mathcal{C}]_7$, and $\pi \mathcal{X} \in [\mathcal{C}]_{7'}$. Then there exists a subfamily $\mathcal{X}' \subset \mathcal{X}$, with $1 + \operatorname{card} \mathcal{X}' \leq c(\pi \mathcal{X})$, such that different components of $\pi \mathcal{X}$ are contained in different components of $\pi \mathcal{X}'$; in particular, $c(\pi \mathcal{X}') \geq c(\pi \mathcal{X})$.

A result of Helly’s type for members of $[\mathcal{C}]_2$ is given by

**Theorem 3.** Let $\mathcal{C}$ be $\gamma$-intersectional and $\mathcal{N}_0$-nonadditive, with the Helly property of order $h$ and limit $\gamma^*$, $\gamma^* \geq \mathcal{N}_0 > h$. Let $\mathcal{X} \subset [\mathcal{C}]_2$ be such that card $\mathcal{X} \leq \gamma + 1$ and $K' \cap K'' \in [\mathcal{C}]_2$ for all $K', K'' \in \mathcal{X}$. Then $\mathcal{X}$ has the Helly property of order $2h$ with limit $\gamma^*$.

3. **Proofs.**

**Proof of Theorem 1.** Obviously

$$K = \sum_{a' \in A'; a'' \in A''} (C_{a'} \cap C_{a''})$$

is a decomposition of $K$. If for each $a' \in A'$ and each $a'' \in A''$ either $C_{a'} \cap C_{a''} = \emptyset$ or $C_{a'} \cap C_{a''} = C_{a'}$, the theorem is proved. Assume on the contrary that there exists an $a'_0 \in A'$ and an $a''_0 \in A''$ such that $C_{a'_0} \cap C_{a''_0}$ is neither $\emptyset$ nor $C_{a'_0}$. Let $A'_0 = \{a'' \in A'': C_{a'_0} \cap C_{a''} \neq \emptyset\}$. Then $2 \leq \operatorname{card} A'_0 < \gamma + 1$ and $C_{a'_0} = C_{a'_0} \cap K = C_{a'_0} \cap \sum_{a'' \in A'} C_{a''} = \sum_{a'' \in A'} (C_{a''} \cap C_{a'_0})$, in contradiction to the $\gamma$-nonadditivity of $\mathcal{C}$.

**Proof of Theorem 2.** (i) Let $c(\pi \mathcal{X}) \geq 2$. Then there exist points $x_1$ and $x_2$ contained in different components $C^* \subset C^* \subset K = \pi \mathcal{X}$. For some $K_0 \in \mathcal{X}$ the points $x_1$ and $x_2$ are contained in different com-
ponents of $K_0$; indeed, otherwise there would for each $K \in \mathcal{K}$ exist a component $C'$ of $K$ with $x_1, x_2 \in C'$. Now $C = \pi \{ C': K \in \mathcal{K} \} \in \mathcal{C}$ but, on the other hand, $C = (C \cap K^*) \cup (C \cap C'_x) + (C \cap C'_z)$, and none of the components is empty (since $x_i \in C \cap C'_i$), contradicting the $\gamma'$-non-additivity of $\mathcal{C}$. If $c(K^*) = 2$, it follows at once from the corollary to Theorem 1 that different components of $K^*$ are contained in different components of $K_0$.

(ii) We now assume that $c(K^*) = n$ is finite, $n > 2$, and that the theorem is proved for all $n'$ with $n' < n$. We start as in (i) with a set $K_0 = \sum_{v \in N} C_v \in \mathcal{K}$, where $N = c(K_0) \geq 2$, such that $C_v \cap K^* \neq \emptyset$ and $C_v \cap K^* \neq \emptyset$. Let $q_v = c(K^* \cap C_v) = 0$ for $v \in N$. By Theorem 1 we have

\[ \sum_{v \in N} q_v = c(K^*) = n. \]

This implies that $N_0 = \{ v \in N : q_v > 0 \}$ is finite and contains at most $n$ elements. Let us assume that $N_0 = \{ 1, 2, \ldots, t \}$ and that the components of $K_0$ are labeled in such a way that $q_v \geq 2$ for $1 \leq v \leq s$, and $q_v = 1$ for $s < v \leq t$. If $s = 0$, then (*) implies $t = n$, and by the corollary to Theorem 1 the $n$ components of $K_0$ contain the $n$ components of $K^*$, as claimed. Thus we are left with the case $s \geq 1$; then $2 \leq t < n$, $q_1 \geq 2$ and, by the choice of $K_0$, $q_2 \geq 1$; therefore, by (*), $q_v < n$ for all $v \in N_0$. This allows us to apply the inductive assumption to each of the $s$ families $\mathcal{K}_v = \{ C_v \cap K : K \in \mathcal{K} \}, 1 \leq v \leq s$. It follows that for each $v$, with $1 \leq v \leq s$, there exists a subfamily $\mathcal{K}_v' \subset \mathcal{K}_v$, containing $p_v \leq q_v - 1$ members, such that the different components of $C_v \cap K^*$ are contained in different components of $\pi \mathcal{K}_v'$. The family $\mathcal{K}' = \{ K_0 \}

\cup (U_{v=1} \mathcal{K}_v')$ satisfies all the conditions of the theorem. Indeed, different components of $K^*$ are, by the corollary to Theorem 1, contained in different components of $\pi \mathcal{K}'$; but on the other hand, $\mathcal{K}'$ contains only $1 + \sum_{v=1}^s p_v \leq 1 - s + \sum_{v=1}^s q_v = 1 - s + n - (t - s) = n + 1 - t \leq n - 1 < c(K^*)$ members.

(iii) There remains the case in which $k = c(K^*)$ is infinite. Let $\omega$ be the initial ordinal of $k$ and let $K^* = \pi \mathcal{K} = \sum_{\mu \leq \omega} C^*_. \mu$. For each $v < \omega$ let $x_v \in C^*_. \mu$ such that $x_v$ and $x_\mu$ are contained in different components of $K^*$. Let $\mathcal{K}' = \{ K^*_. \mu : x_\mu < \omega \}$. Then $\text{card } \mathcal{K}' \leq (\text{card } \omega)^2 = k$. For the family $\mathcal{K}'$ we have $c(\pi \mathcal{K}') \geq k$ since $x_v$ and $x_\mu$ belong to different components of $\pi \mathcal{K}'$. By an argument similar to that used in the proof of Theorem 1 it follows that different components of $K^*$ are contained in different components of $\pi \mathcal{K}'$. This ends the proof of Theorem 2.
Proof of Theorem 3. For some fixed \( h \) assume the theorem false; let \( k \) be the minimal cardinal for which there exists a family with \( \mathcal{K} = k \) contradicting the theorem.

(i) Assume \( k \) finite. Then for each \( K^* \in \mathcal{K} \) we have 
\[
\pi\{K \in \mathcal{K} : K \neq K^*\} \neq \emptyset.
\]
Let \( \mathcal{K}_i = \{K \in \mathcal{K} : c(K) = i\} \) for \( i = 1, 2 \), and let \( K = C_1 + C_2 \) for all \( K \in \mathcal{K}_2 \). We assume that \( \mathcal{K} \) is chosen in such a way that card \( \mathcal{K}_1 \) + 2 card \( \mathcal{K}_2 \) (the total number of components of members of \( \mathcal{K} \)) is minimal. This implies that for each \( K' \in \mathcal{K}_2 \) and \( i = 1, 2 \), there exists a \( K^0 = K^0(C'_i) = K^0(K', i) \in \mathcal{K} \) such that 
\[
\pi\{K \in \mathcal{K} : K \neq K^0\} \subset C'_i.
\]
We shall show that each \( C'_i \cap K \neq \emptyset \) for all \( K' \in \mathcal{K}_2, K \in \mathcal{K} \), and \( i = 1, 2 \). Let us assume, to the contrary, that there exists \( P' \in \mathcal{K}_2, P \in \mathcal{K} \) and \( i = 1 \) or 2 such that \( C'_i \cap K = \emptyset \). (Without loss of generality we shall assume \( i = 1 \).) Since \( \emptyset \neq \pi\{K \in \mathcal{K} : K \neq K^0(K', 1)\} \subset C'_1 \), it follows that \( K_0 = K^0 = K^0(K', 1) \). Then \( C'_i \cap K \neq \emptyset \) for all \( K \neq K^0 \); also \( C'_i \cap K \neq \emptyset \) for all \( K \in \mathcal{K} \), since otherwise \( K' \cap K \cap K^0 \subset (C'_i \cap K^0) \cup (C'_i \cap K) = \emptyset \) would contradict the assumption that any \( 3 < 4 \leq 2h \) members of \( \mathcal{K} \) have a nonempty intersection. Therefore, for each \( K \neq K^0 \), \( c(K' \cap K) = 2 \); hence, for some component \( C_j \) of \( K \) we have \( K \cap C'_j = C_j \cap C'_j \). Now
\[
\pi\{C_j : K \in \mathcal{K}, K \neq K^0\} = C'_1 \cap \pi\{C_j : K \in \mathcal{K}, K \neq K^0\}
\]
\[
= C'_1 \cap \pi\{K \in \mathcal{K} : K \neq K^0\} \subset C'_1 \cap C'_1 = \emptyset.
\]
Since \( \mathcal{E} \) has the Helly property of order \( h \) it follows that for some subset \( \mathcal{K}_0 \) of \( \mathcal{K} \), such that \( K^0 \in \mathcal{K}_0 \) and with card \( \mathcal{K}_0 = h_0 \leq h \), we have 
\[
\pi\{C_j : K \in \mathcal{K}_0\} = \emptyset.
\]
For the family \( \mathcal{K}^* = \{K', K^0\} \cup \mathcal{K}_0 \) we have therefore 
\[
\pi \mathcal{K}^* \subset (C'_i \cap K') \cup (C'_i \cap \pi \mathcal{K}_0) = \emptyset,
\]
although card \( \mathcal{K}^* \leq h_0 + 2 \leq h + 2 \leq 2h \). This contradiction establishes our assertion.

Next, let \( K^* \in \mathcal{K}_2 \) be chosen arbitrarily. For each \( K \in \mathcal{K}_2 \) it follows from the above and from \( c(K' \cap K) \leq 2 \) that \( c(K' \cap K) = 2 \) and that different components of \( K \) intersect different components of \( K^* \). Let the components of \( K \) be re-labeled, if necessary, in such a way that \( C_i^* \cap C_i \neq \emptyset \) for \( i = 1, 2 \). We claim that for all \( K' \), \( K'' \in \mathcal{K}_2 \) we have \( C'_i \cap C'_i' = \emptyset \), \( i = 1, 2 \). Indeed, otherwise we would have (since each component of one set intersects every other set), \( C'_i \cap C'_i' = C'_i \cap C'_i' = \emptyset \), and therefore \( K^* \cap K' \cap K'' = \emptyset \), which is impossible. Thus, for any \( K', K'' \in \mathcal{K}_2 \),
\[
C'_i \cap C'_j' = \emptyset \quad \text{if} \quad i \neq j
\]
\[
\neq \emptyset \quad \text{if} \quad i = j.
\]
Now we consider the families \( \mathcal{E}_i = \mathcal{K}_1 \cup \{C_i : K \in \mathcal{K}_2\} \) for \( i = 1, 2 \). The assumption \( \pi \mathcal{E} = \emptyset \) implies that \( \pi \mathcal{E}_i = \emptyset \) for \( i = 1, 2 \). Since \( \mathcal{E}_i \subset \mathcal{E} \),
there exist $h$ or less members of $\mathcal{C}_i$ whose intersection is empty, $i=1, 2$. But then the intersection of the corresponding members of $\mathcal{K}$ is also empty, although it involves at most $2h$ members of $\mathcal{K}$. The contradiction reached proves the theorem for finite $k$.

(ii) Let $k$ be infinite, $k<\gamma^*$, and the theorem true for all families with less than $k$ members. Let $\omega$ be the initial ordinal of $k$, let $A$ be the set of ordinals $A=\{\alpha: \alpha<\omega\}$, and let $\mathcal{K}=\{K_\alpha: \alpha<\omega\}$. By the inductive assumption we have $\bigcap_{\alpha<\omega}K_\alpha\neq\emptyset$ for each $\mu<\omega$. If for some $K_\alpha$ one of its components does not intersect some $K_\beta$, we omit this component and take the other component to be the new $K_\alpha$. By the inductive assumption, the new $K_\alpha$ satisfy $\bigcap_{\alpha<\omega}K_\alpha\neq\emptyset$ for all $\mu<\omega$. From here on we proceed as in the final part of (i): we re-label (if necessary) the components of some $K_\alpha$ with $c(K_\alpha)=2$, construct the families $\mathcal{C}_i$ and derive a contradiction from the assumption that $\bigcap_{\alpha<\omega}K_\alpha=\emptyset$. This terminates the proof of Theorem 3.

4. Remarks. 1. Theorem 2 fails if card $\pi\mathcal{K}$ is infinite and $\mathcal{K}'$ is assumed to satisfy card $\mathcal{K}'<\text{card } \pi\mathcal{K}$. E.g., starting from the family $\mathcal{S}$ (preceding Definition 3), with card $\omega=k>\aleph_0=\text{card } \omega_0$, let $\mathcal{K}=\{[\omega_0, \alpha] \cup [\alpha+\omega_0, \omega]: \alpha \text{ limit ordinal } <\omega\}$. Then $c(\pi\mathcal{K})=k$, but the intersection of any $k'<k$ members of $\mathcal{K}$ has only $k'$ components. Similar examples are easily found for $c(\pi\mathcal{K})=\aleph_0$.

2. Probably the most interesting immediate application of Theorem 3 is to convex sets in $E^d$. To satisfy the condition of nonadditivity we may consider, e.g., families consisting only of closed (or only of open) convex sets. The following example shows that Theorem 3 does not hold if $\mathcal{C}$ is, e.g., the family of all convex sets in $E^2$. (Simple examples of a similar nature show the necessity of nonadditivity assumptions in Theorem 2.) Let $D$ denote a closed disc with center 0. Let $K_0$ be obtained from $D$ by deleting 0. Let $x_i$, $i=1, 2, \ldots, 6$, be equidistant points on the boundary of $D$, $(x_i=x_{i+6})$. For each $i$, $1\leq i \leq 6$, let $K_i$ be obtained from $D$ by deleting the open small arc of $\text{Bd } D$ determined by $x_{i-1}$ and $x_{i+1}$, and the open sector determined by these two points and 0. Then each $K_i$, $0\leq i \leq 6$, as well as the intersection of any two $K_i$, is the disjoint union of two convex sets, and any six $K_i$ have a nonempty intersection. Nevertheless, $\bigcap_{i=0}^6K_i=\emptyset$. As is easily verified, the same reasoning applies to the case where 7 or 8 equidistant points are chosen on $\text{Bd } D$. We conjecture that for the family of all convex sets in $E^2$ a result analogous to Theorem 3 holds, with 9 instead of $2h$.

3. The following statement (with obvious refinements) is conjectured: If $\mathcal{C}$ is an intersectional and nonadditive family with un-
limited Helly property of order \( h \) and if \( \mathcal{K} \subseteq [\mathcal{C}]_n \) is such that the intersection of any 2, 3, \( \cdots \), \( n \) members of \( \mathcal{K} \) also belongs to \( [\mathcal{C}]_n \), then \( \mathcal{K} \) has the unlimited Helly property of order \( nh \). Simple examples show that \( nh - 1 \) may not be substituted for \( nh \) in this conjecture. If \( \mathcal{C} \) is the family of segments in \( E^1 \), the conjecture is easily provable.

4. Let \( \mathcal{C}^{(n)} \) denote the family of all compact, convex subsets of \( E^n \); in [1], a function \( \Delta(K) \), with \( 0 \leq \Delta(K) \leq +\infty \), was defined for all compact sets \( K \subseteq E^n \) in such a way that \( \Delta(K) < \infty \) if and only if \( K \subseteq [\mathcal{C}^{(n)}]_n \). Theorem 2 of [1] may be formulated as follows: For any finite \( n \geq 1 \) and real \( d < \infty \) there exists a finite \( h = h(n, d) \) such that the family \( \{ K \subseteq [\mathcal{C}^{(n)}]_n; \Delta(K) \leq d \} \) has the unlimited Helly property of order \( h \). By applying the methods of [1] it may be shown that for each finite \( n \geq 1 \) and \( d < \infty \) there exists a finite \( k = k(n, d) \) such that \( \Delta(K) \leq d \) implies \( K \subseteq [\mathcal{C}^{(n)}]_k \).

References


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