SUBORDINATING FACTOR SEQUENCES FOR CONVEX MAPS OF THE UNIT CIRCLE

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I. **Introduction.** Let K denote the class of functions

$$f(z) = \sum_{\nu=1}^{\infty} a_\nu z^\nu \tag{1}$$

which are regular in the unit circle and map it onto a schlicht convex domain. In a recent paper [1] Pólya and Schoenberg have shown that in order for $f(z) \in K$ it is necessary and sufficient that each of the functions

$$V_n(z; f) = \frac{1}{C_{2n,n}} \sum_{\nu=1}^{n} C_{2n,n+\nu} a_\nu z^\nu \qquad (n = 1, 2, \cdots) \tag{2}$$

belong to K.

We will use the notation

$$f(z) \subseteq g(z) \tag{3}$$

("$f(z)$ is subordinate to $g(z)$") to mean that $f(z)$, $g(z)$ are both regular in $|z| < 1$, that $g(z)$ is univalent there, and that every value taken by $f(z)$ in $|z| < 1$ is also taken by $g(z)$ (see [2; 3]). It was shown in [1] that

$$V_n(z; f) \subseteq f(z) \qquad (n = 1, 2, \cdots) \tag{4}$$

for every $f(z) \in K$, and it was pointed out that even

$$V_1(z; f) \subseteq V_2(z; f) \subseteq \cdots \subseteq f(z) \tag{5}$$

is likely, though this was not verified except for

$$f_0(z) = z(1 - z)^{-1}. \tag{6}$$

In the following paragraphs we will show how consideration of the problem (5) leads, in a natural way, to the question of characterizing certain kinds of factor sequences (see [6]), and although we cannot decide the truth or falsity of (5), a closely related question will be completely settled (Theorem 2, *infra*).

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II. **Subordinating factor sequences.** An infinite sequence $\{b_\nu\}_1^\infty$ of complex numbers will be called a subordinating factor sequence if whenever

$$f(z) = \sum_{\nu=1}^{\infty} a_\nu z^\nu \in K \tag{7}$$

we have

$$\sum_{\nu=1}^{\infty} a_\nu b_\nu z^\nu \subseteq f(z). \tag{8}$$

A finite sequence $\{b_\nu\}_1^n$ will be called a subordinating factor sequence if (7) implies (8) whenever $a_{n+1}=a_{n+2}= \cdots =0$. The class of such infinite sequences we denote by $\mathfrak{F}$, and that of sequences of length n by $\mathfrak{F}_n$.

THEOREM 1. *The proposition*

$$\left\{1 - \frac{\nu^2}{n^2}\right\}_{\nu=1}^{n} \in \mathfrak{F}_n \tag{9}$$

implies (5).

PROOF. This is immediate from the easily established identity

$$\left(1 + \frac{z}{n}\frac{d}{dz}\right)\left(1 - \frac{z}{n}\frac{d}{dz}\right) V_n(z;f) = V_{n-1}(z;f) \quad (n = 2, 3, \cdots) \tag{10}$$

and the definition of $\mathfrak{F}_n$.

We do not know how to characterize sequences of $\mathfrak{F}_n$. The following result, however, completely describes the class $\mathfrak{F}$.

THEOREM 2. *The following three properties of a sequence of complex numbers are equivalent:*

(I) $\{b_\nu\}_1^\infty \in \mathfrak{F}$;

(II) $\operatorname{Re}\left\{1 + 2\sum_{\nu=1}^{\infty} b_\nu z^\nu\right\} > 0 \qquad (|z| < 1)$;

(III) $b_\nu = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{i\nu\theta} d\psi(\theta) \qquad (\nu = 0, 1, 2, \cdots; b_0 = 1; \psi(\theta)\uparrow)$.

PROOF. The equivalence of (II) and (III) is classical. Now suppose (I) holds. Then

$$\sum_{\nu=1}^{\infty} b_\nu z^\nu \subseteq \sum_{\nu=1}^{\infty} z^\nu = z(1-z)^{-1},$$

which is to say that,

$$\operatorname{Re}\left\{\sum_{\nu=1}^{\infty} b_\nu z^\nu\right\} > -\frac{1}{2} \qquad (|z| < 1),$$

which proves (II). Conversely, if (III) holds, let

$$f(z) = \sum_{\nu=1}^{\infty} a_\nu z^\nu \in K. \tag{11}$$

Then

$$\begin{aligned} \sum_{\nu=1}^{\infty} a_\nu b_\nu z^\nu &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{\nu=1}^{\infty} a_\nu e^{i\nu\theta} r^\nu e^{i\nu\phi} d\psi(\theta) \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(re^{i(\theta+\phi)}) d\psi(\theta). \end{aligned} \tag{12}$$

The left hand side is thus exhibited as the centroid of a nonnegative mass distribution of total mass one, on a convex curve, and therefore lies inside that curve, which was to be shown.

Several results, some well known, follow immediately from Theorem 2.

COROLLARY 1. *If* $\{b_\nu\}_1^\infty \in \mathfrak{F}$, $\{c_\nu\}_1^\infty \in \mathfrak{F}$, *then* $\{b_\nu c_\nu\}_1^\infty \in \mathfrak{F}$.

Since from (12), the result of applying a sequence of $\mathfrak{F}$ to an arbitrary analytic function is a function which maps the unit circle into the convex hull of the original image, the result of applying these two sequences to $f(z)$ in succession is clearly subordinate to $f(z)$.

COROLLARY 2. *If*

$$\operatorname{Re}\left\{1 + 2\sum_1^{\infty} a_\nu z^\nu\right\} > 0, \quad \operatorname{Re}\left\{1 + 2\sum_1^{\infty} b_\nu z^\nu\right\} > 0$$

then

$$\operatorname{Re}\left\{1 + 2\sum_1^{\infty} a_\nu b_\nu z^\nu\right\} > 0 \qquad (|z| < 1).$$

This well-known result [5, VII, 43] is clear from Theorem 2 and Corollary 1.

COROLLARY 3. *The image of the unit circle under the mapping*

$$f(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu$$

of K, contains the circle $|W| < 1/2$, *the constant being sharp.*

This result, due to Study [4] (compare [2, p. 223]; [1, p. 320]) is precisely the assertion that the sequence $1/2, 0, 0, \cdots$ belongs to $\mathfrak{F}$, which is obvious from Theorem 2, (II). The sharpness is shown, as usual, by the example (6).

COROLLARY 4. *Equation* (4) *is true.*

Indeed, from (6) and (6′) of [1] with $z = e^{i\theta}$, there follows

$$\begin{aligned}\operatorname{Re}\left\{1 + 2V_n\left(z; \frac{z}{1-z}\right)\right\} &= 1 + 2\sum_{\nu=1}^{n} \frac{n!}{(n-\nu)!}\,\frac{n!}{(n+\nu)!}\cos \nu\theta \\ &= \frac{(n!)^2}{(2n)!}\left(2\cos\frac{\theta}{2}\right)^{2n} \\ &\geqq 0\end{aligned}$$

whence the sequence

$$b_\nu = \begin{cases} C_{2n,n+\nu}/C_{2n,n} & (\nu = 1, 2, \cdots, n) \\ 0 & (\nu \geqq n+1) \end{cases} \tag{14}$$

belongs to $\mathfrak{F}$, which is exactly what (4) asserts (our proof is really identical with that in [1]).

COROLLARY 5. *Let the functions* $f(z) = \sum_1^\infty a_\nu z^\nu$, $g(z) = \sum_1^\infty b_\nu z^\nu$ *belong to K, and map* $|z| < 1$ *onto domains* $\mathfrak{D}'$, $\mathfrak{D}''$, *respectively, both contained in* $\operatorname{Re} w > -1/2$. *Then the function* $\sum_1^\infty a_\nu b_\nu z^\nu$ maps $|z| < 1$ onto *a domain* $\mathfrak{D} \subseteq \mathfrak{D}' \cap \mathfrak{D}''$.

This result, which is related to a conjecture of Pólya-Schoenberg on the Hadamard product of functions of K, follows by noting that $\{a_\nu\}_1^\infty \in \mathfrak{F} \Rightarrow \mathfrak{D} \subseteq \mathfrak{D}''$, and $\{b_\nu\}_1^\infty \in \mathfrak{F} \Rightarrow \mathfrak{D} \subseteq \mathfrak{D}'$ which was to be shown.

Concerning the open question (9) we may now see that, in any event, the sequence $\{1 - \nu^2/n^2\}_1^n$ is not extendable to a sequence of $\mathfrak{F}$ since that would require the positivity of the Toeplitz matrix

$$T = \begin{pmatrix} 1 & 1 - 1/n^2 & 1 - 4/n^2 & \cdots \\ 1 - 1/n^2 & 1 & 1 - 1/n^2 & \cdots \\ 1 - 4/n^2 & 1 - 1/n^2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \tag{15}$$

whereas the 3×3 determinant in the upper left corner has the value $-8n^{-6}<0$.

III. **On the composition of convex maps.** In [1] it was conjectured that if $f(z)=\sum a_\nu z^\nu \in K$ and $g(z)=\sum b_\nu z^\nu \in K$ then so does $h(z) = \sum a_\nu b_\nu z^\nu$. We state here a single proposition whose truth would imply both this conjecture and (5) at once. It is

PROPOSITION 1. *The coefficients of a convex function preserve subordination between convex functions. That is, if* $\sum a_\nu z^\nu$, $\sum b_\nu z^\nu$, $\sum c_\nu z^\nu$ *are all in* K, *and if*

$$\sum a_\nu z^\nu \subseteq \sum b_\nu z^\nu$$

then

$$\sum a_\nu c_\nu z^\nu \subseteq \sum b_\nu c_\nu z^\nu.$$

Indeed, if this is true, then since (5) holds in the case (6) it holds in general. Further, by applying the sequence $\{b_\nu\}_1^\infty$ to the relations

$$V_n(z;f) \subseteq f(z) \qquad (n = 1, 2, \cdots)$$

we would find that the means of the function $\sum a_\nu b_\nu z^\nu$ are subordinate to the function itself, and in view of a recent result of Robertson [7], it would follow that $\sum a_\nu b_\nu z^\nu \in K$.

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