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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

THE GEÖCZE k-AREA AND A CYLINDRICAL PROPERTY

TOGO NISHIURA

In [5], a definition of the Geöcze k-area of a mapping from admissible sets of Euclidean k-space E_k into Euclidean n-space E_n $(2 \leqq k \leqq n)$ is given. This definition is an extension of the Geöcze area given in [3]. With this definition of Geöcze k-area, a treatment of Geöcze k-area is developed for flat mappings $(k=n)$ paralleling the treatment of Geöcze area for plane mappings given in [3]. The present paper gives results concerning the Geöcze k-area for mappings from admissible sets of E_k into E_n $(n>k\geqq 2)$. A cylindrical property is defined for mappings in harmony with [3, (16.10)]. This property, which has had an essential part in the proofs of the main theorems for Lebesgue area for mappings from admissible sets of E_2 into E_n [3] and which has been used in other research, is shown to play a prominent role in the extension of the theory of Geöcze area to higher dimensions. An example is given to show that the theorems concerning the cylindrical property in [3] are no longer valid for $k\geqq 3$. These theorems are shown to be valid under a certain restrictive hypothesis found in the literature.

1. Notations and definitions. If X is a set in E_k, then $\overline{X}$, X^0, and X^* will denote respectively the closure, interior, and boundary of X.

A *polyhedral region* R in E_k is the point-set covered by a strongly connected k-complex situated in E_k. A polyhedral region R is called *simple* if $E_k - R$ is connected (see [5]).

By a *figure* F we mean a finite union of nonoverlapping polyhedral regions in E_k such that the interior of the union is the union of the interior of the finitely many polyhedral regions. A set A in E_k is said to be *admissible* in each of the following cases:

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(a) A is an open set in E_k.

(b) A is a figure.

(c) A is a homeomorphic image of a figure.

(d) A is a set open in the type (b) or (c) above.

By $\mathfrak{J}(k, n)$ we mean the set of all continuous mappings (T, A) from admissible sets A in E_k into E_n $(2 \leqq k \leqq n)$. A continuous mapping (T, A) is called *flat* if $(T, A) \in \mathfrak{J}(k, k)$.

Let (T_0, A_0), (T_j, A_j) $(j=1, 2, \cdots)$ be a sequence of mappings in $\mathfrak{J}(k, n)$. (T_j, A_j) $(j=1, 2, \cdots)$ is said to *converge* to (T_0, A_0) if

(i) $A_j \subset A_{j+1} \subset A_0$ for all j;

(ii) $\bigcup_{j=1}^{\infty} A_j^0 = A_0^0$;

(iii) $\operatorname{Sup}\,[\,|T(w) - T_j(w)| : w \in A_j] \to 0$ as $j \to \infty$.

If $k < n$ are positive integers then Ω_n^k is the set of all k-termed sequences $\xi = (\xi_1, \xi_2, \cdots, \xi_k)$ of integers such that $1 \leqq \xi_1 < \xi_2 < \cdots < \xi_k \leqq n$. We shall assume that Ω_n^k is lexicographically ordered.

If $x = (x^1, x^2, \cdots, x^n) \in E_n$ then by the mapping P_n^ξ, $\xi \in \Omega_n^k$, we shall mean the projection of E_n onto the coordinate hyperspace E_k^ξ where $P_n^\xi(x) = (x^{\xi_1}, x^{\xi_2}, \cdots, x^{\xi_k}) \in E_k^\xi$.

If $(T, A) \in \mathfrak{J}(k, n)$ then by (T^ξ, A) we mean the mapping $(P_n^\xi T, A)$, $\xi \in \Omega_n^m$, $k \leqq m \leqq n$. If $m = k$ then (T^ξ, A), $\xi \in \Omega_n^k$, forms a collection of C_k^n flat mappings from the admissible set A into $E_k^\xi \subset E_n$, $\xi \in \Omega_n^k$.

Let $(T, A) \in \mathfrak{J}(k, n)$ and D be any bounded open connected set such that $\overline{D} \subset A$. If $C^\xi: (T^\xi, D^*)$, $\xi \in \Omega_n^k$, then a topological index $O(x; C^\xi)$, $x \in E_k^\xi$, is defined $\{$see [5, (1.3)] and also [6; p. 125]$\}$. The function $O(x; C^\xi)$ is a measurable function of $x \in E_k^\xi$ and the integral $v(T^\xi, D) = (E_k^\xi)\int |O(x; C^\xi)|$ exists (finite or infinite) for each $\xi \in \Omega_n^k$. $[v(T^\xi, D): \xi \in \Omega_n^k]$ forms a $1 \times C_k^n$ vector. Denote by $v(T, D)$ the Euclidean norm of this vector; i.e., $v(T, D) = [\sum v^2(T^\xi, D)]^{1/2}$, where $\sum$ ranges over $\xi \in \Omega_n^k$. Denote by $\mathfrak{S}$ a finite system of nonoverlapping simple polyhedral regions $\pi \subset A$. We define the *Geöcze k-area* of (T, A) to be the number $V(T, A) = \operatorname{Sup} \sum v(T, \pi)$, where $v(T, \pi) = v(T, \pi^0)$, $\sum$ ranges over $\pi \in \mathfrak{S}$ and the supremum is taken over all possible systems $\mathfrak{S}$. If $V(T, A) < \infty$, then the integral $u(T^\xi, D) = (E_k^\xi)\int O(x; C^\xi)$ exists for each $\xi \in \Omega_n^k$. Let $u(T, D) = [\sum u^2(T^\xi, D)]^{1/2}$, and $U(T, A) = \operatorname{Sup} \sum u(T, \pi)$, where $u(T, \pi) = u(T, \pi^0)$ and the sum $\sum$ and the supremum are taken as above.

Let $(T, A) \in \mathfrak{J}(k, n)$. Then $\Gamma(T, A)$ will denote the collection of all maximal connected sets of A on which (T, A) is constant. If A is compact then $\Gamma(T, A)$ forms an upper semicontinuous decomposition of A.

2. **The cylindrical property.** In this section we confine ourselves to continuous mappings $(T, A) \in \mathfrak{J}(k, n)$ with A compact and $n > k$. For

such mappings we have C_k^n flat mappings (T^ξ, A), $\xi \in \Omega_n^k$. Associated with these mappings are their corresponding upper semicontinuous decompositions $\Gamma(T, A)$, $\Gamma(T^\xi, A)$, $\xi \in \Omega_n^k$, of A. A point $x \in E_k^\xi$ is said to have the *cylindrical property* with respect to (T, A) if (i) $x \in T^\xi(A)$; (ii) there is a $g \in \Gamma(T^\xi, A)$ such that $T^\xi(g) = x$ and $T(g)$ is nondegenerate.

2.1. *Theorems concerning the cylindrical property.* For each $\xi \in \Omega_n^k$ let X^ξ be the set of all $x \in E_k^\xi$ having the cylindrical property with respect to (T, A).

THEOREM (i). *Let $(T, A) \in \mathfrak{J}(2, n)$ with A compact and $V(T, A) < \infty$. Then, for each $\xi \in \Omega_n^2$, X^ξ has zero Lebesgue 2-measure.*

PROOF. This theorem was first proved by L. Cesari in [2] for the case $n = 3$. It is easily shown that this special case implies the theorem. Another proof is also given in [4, Theorem 8.12].

Theorem (i) above shows that for $k = 2$, the condition $V(T, A)$ is finite has very strong implications, namely, the set X^ξ has zero measure for all $\xi \in \Omega_n^2$. This no longer is true when $k > 2$ as an example in (2.2) below shows. The following theorem gives a sufficient condition on the mapping $(T, A) \in \mathfrak{J}(k, n)$ to make the Lebesgue k-measure of X^ξ zero, $\xi \in \Omega_n^k$, $n > k \geqq 2$.

THEOREM (ii). *Let $(T, A) \in \mathfrak{J}(k, n)$ $(2 \leqq k < n)$, A be compact and $H_n^{k+1}[T(A)] = 0$, where $H_n^{k+1}[\cdot]$ denotes the $(k+1)$-dimensional Hausdorff measure in E_n. Then $m_k[X^\xi] = 0$ for all $\xi \in \Omega_n^k$, where $m_k[\cdot]$ is the k-dimensional Lebesgue measure.*

PROOF. Let $\alpha \in \Omega_n^{k+1}$, P_n^α and E_{k+1}^α be defined as in §1 above. Let us fix a $\xi \in \Omega_n^k$. For each $\alpha \in \Omega_n^{k+1}$ with $E_k^\xi \subset E_{k+1}^\alpha$, let X_α^ξ denote the set of all points of E_k^ξ which have the cylindrical property with respect to the mapping $(P_n^\alpha T, A)$. Then we have that $X^\xi = \bigcup X_\alpha^\xi$, where $\bigcup$ ranges over all $\alpha \in \Omega_n^{k+1}$ with $E_k^\xi \subset E_{k+1}^\alpha$. Hence it is sufficient to show that $m_k[X_\alpha^\xi] = 0$ for each $\alpha \in \Omega_n^{k+1}$ with $E_k^\xi \subset E_{k+1}^\alpha$.

Since $0 = H_n^{k+1}[T(A)] \geqq H_{k+1}^{k+1}[P_n^\alpha T(A)] = m_{k+1}[P_n^\alpha T(A)] \geqq 0$, $\alpha \in \Omega_n^{k+1}$, we have $m_{k+1}[P_n^\alpha T(A)] = 0$ for each $\alpha \in \Omega_n^{k+1}$. This implies that we need only study the case where (T, A) is a continuous mapping of the compact admissible set $A \subset E_k$ into E_{k+1} with $m_{k+1}[T(A)] = 0$. Suppose this is the case and let $\xi \in \Omega_{k+1}^k$. For the sake of simplicity, let ξ be such that $E_k^\xi = \{x \in E_{k+1} : x^1 = 0\}$. Let $[a_i, b_i]$, $i = 1, 2, \cdots$, be the countable collection of closed intervals on the x^1-axis of E_{k+1}, where a_i and b_i are rational numbers, $a_i < b_i$. Denote by P the usual projection of E_{k+1} onto the x^1-axis. If X_i^ξ denotes the set of all $x \in X^\xi$ with the property that $(T^\xi)^{-1}(x)$ has a component g such that $PT(g)$

covers $[a_i, b_i]$, then $X^\xi = \bigcup_{i=1}^\infty X_i^\xi$. Since A is compact and $\Gamma(T^\xi, A)$ is an upper semicontinuous decomposition of A, X_i^ξ is closed and hence compact for each i.

Let $Z_i = T(A) \cap (P_{k+1}^\xi)^{-1}(X_i^\xi)$. Z_i is compact and hence measurable. Hence for each i, $0 = m_{k+1}[T(A)] \geqq m_{k+1}[Z_i] \geqq m_k[X_i^\xi] \cdot (b_i - a_i) \geqq 0$. Therefore $m_k[X_i^\xi] = 0$. Thus we conclude that $m_k[X^\xi] = 0$ and the theorem is proved.

2.2. EXAMPLE. We give an example to show that in (2.1.ii) above, the condition $H_n^{k+1}[T(A)] = 0$ cannot be replaced by $V(T, A) < \infty$ if $n > k = 3$. This example will be used again in the next section.

Let $A = \{w \in E_3 : 0 \leqq w^i \leqq 1,\ i = 1, 2, 3\}$, $A_0 = \{w \in A : w^3 = 0\}$, and let P_0 be the usual projection map of A onto A_0. To facilitate discussion, we will denote the m-dimensional hyperspace of E_n spanned by $(x^1, x^2, \cdots, x^m, 0, \cdots, 0)$ by $E(x^1, x^2, \cdots, x^m)$.

Let $(T_0, A_0): x^i = f^i(w^1, w^2)$, $w \in A_0$, $i = 1, 2, 3$, be the continuous mapping defined by A. Besicovitch in [1] of the square A_0 into $E(x^1, x^2, x^3)$ so that (T_0, A_0) is a homeomorphism, $m_3[T_0(A_0)] > 0$ and the 2-area of (T_0, A_0) is a positive number ϵ. Let (T, A) be defined by $x^i = f^i(w^1, w^2)$, $i = 1, 2, 3$, $x^4 = w^3$, $w = (w^1, w^2, w^3) \in A$. Clearly (T, A) is a homeomorphism. Hence every element of $\Gamma(T, A)$ is a point $w \in A$. Let $\xi \in \Omega_4^3$ be such that $E_3^\xi = E(x^1, x^2, x^3)$. Then every element $g \in \Gamma(T^\xi, A)$ is an interval. Hence $X^\xi = T^\xi(A)$, and $m_3[X^\xi] = m_3[T^\xi(A)] = m_3[T_0(A_0)] > 0$, since $T^\xi(w) = T_0 P_0(w)$.

Let us show $V(T, A) \leqq \epsilon$. Since the 2-area of (T_0, A_0) is ϵ, by [3, (24.1.i)], there exists a sequence of quasi-linear mappings $(Q_{0j}, A_0): x^i = h_j^i(w^1, w^2)$, $i = 1, 2, 3$, $w \in A_0$, $j = 1, 2, \cdots$, converging to (T_0, A_0) and the 2-area of (Q_{0j}, A_0) converging to ϵ. Let (Q_j, A) be the quasi-linear mapping $x^i = h_j^i(w^1, w^2)$, $i = 1, 2, 3$, $x^4 = w^3$, $w \in A$, $j = 1, 2, \cdots$. Clearly, $(Q_j, A) \to (T, A)$ as $j \to \infty$. By [5, (2.2.x)], $V(Q_j, A)$ is equal to the 2-area of (Q_{0j}, A_0). Hence by the lower semicontinuity of the Geöcze k-area, $\epsilon = \lim_{j\to\infty} V(Q_j, A) \geqq V(T, A)$ [5, (2.2.vi)]. Thus the example is constructed.

3. **The indices** d, m, σ. In the theory of Geöcze area, the indices d, m, σ play an important role (see [3]). These indices have been defined for the Geöcze k-area [5, (3.3)]. We give these definitions below.

Let $(T, F) \in \mathfrak{J}(k, n)$ where F is a figure. Denote by $\mathfrak{S} = \mathfrak{S}' \cup \mathfrak{S}''$ a finite subdivision of F into nonoverlapping polyhedral regions, where $\mathfrak{S}'$ is the collection of simple polyhedral regions and $\mathfrak{S}''$ is the collection of nonsimple polyhedral regions. Then $d = \max[\operatorname{diam} T(R) : R \in \mathfrak{S}]$. $m = \max[m_k[\bigcup T^\xi(R^*)] : \xi \in \Omega_n^k]$, where $\bigcup$ ranges over all $R \in \mathfrak{S}'$ with $R \subset F^0$. $\sigma = \max[\sigma^\xi : \xi \in \Omega_n^k]$, where $\sigma^\xi = \sum'' \sum^* \operatorname{diam} T^\xi(B_i)$, with $\sum^*$

ranging over all components B_i of R^* and $\sum''$ ranging over all $R \in \mathfrak{S}''$, $R \subset F^0$.

THEOREM (i). *If $k=2$ and $V(T, F) < \infty$, then for every $\epsilon > 0$ there is a subdivision $\mathfrak{S} = \mathfrak{S}' \cup \mathfrak{S}''$ of F with indices d, m, $\sigma < \epsilon$.*

THEOREM (ii). *If $k \geqq 2$ and $H_n^{k+1}[T(F)] = 0$, then for every $\epsilon > 0$ there is a subdivision $\mathfrak{S} = \mathfrak{S}' \cup \mathfrak{S}''$ of F with indices d, m, $\sigma < \epsilon$.*

PROOF. The proofs of Theorems (i) and (ii) are essentially the same as the proof of [**3**, (21.1.i)]. The only change needed is the replacement of [**3**, (16.10.i)] by (2.1.i) and (2.1.ii), respectively.

We observe that in the example of (2.2), $d+m+\sigma > \epsilon_0 > 0$ for an appropriate ϵ_0, and $H_n^{k+1}[T(F)] \neq 0$.

4. Theorems on limits with respect to d, m, σ. Let $(T, A) \in \mathfrak{J}(k, n)$, F_j $(j=1, 2, \cdots)$ be a sequence of figures with $F_j \subset F_{j+1} \subset A$ and $\bigcup_{j=1}^{\infty} F_j^0 = A^0$, and $\mathfrak{S}_j = \mathfrak{S}_j' \cup \mathfrak{S}_j''$ be a subdivision of F_j witn indices d_j, m_j, σ_j $(j=1, 2, \cdots)$.

THEOREM (i). *If $A_0 \subset A$ is an admissible set and $d_j + m_j + \sigma_j \to 0$ as $j \to \infty$, then*

$$\lim_{j\to\infty} \sum^{(i)} v(T, q) = V(T, A_0), \quad \lim_{j\to\infty} \sum^{(i)} v(T^\xi, q) = V(T^\xi, A_0), \quad \xi \in \Omega_n^k,$$

where $\sum^{(i)}$ ranges over all $q \in \mathfrak{S}_j'$, $q \subset A_0$.

If, furthermore, $V(T, A) < \infty$, then

$$\lim_{j\to\infty} \sum^{(i)} u(T, q) = U(T, A_0) = V(T, A_0),$$

$$\lim_{j\to\infty} \sum^{(i)} |u(T^\xi, q)| = U(T^\xi, A_0) = V(T^\xi, A_0), \qquad \xi \in \Omega_n^k,$$

where $\sum^{(i)}$ ranges over all $q \in \mathfrak{S}_j'$, $q \subset A_0$.

PROOF. The proof of (i) is essentially the same as [**3**, (21.2.i), (21.3.i, ii)].

From (3.i, ii) we have sufficient conditions on (T, A) for the existence of subdivisions $\mathfrak{S} = \mathfrak{S}' \cup \mathfrak{S}''$ of $F \subset A$ with arbitrary small indices d, m, σ. Namely, (1) $k=2$ and $V(T, A) < \infty$; or (2) $k \geqq 2$ and $H_n^{k+1}[T(A)] = 0$.

5. In this section we give some theorems concerning the Geöcze k-area of mappings $(T, A) \in \mathfrak{J}(k, n)$.

THEOREM (i). *If $(T, A) \in \mathfrak{J}(k, n)$ and $H_n^{k+1}[T(A)] = 0$, then*

$[\sum V^2(T^\xi, A)]^{1/2} \leqq V(T, A) \leqq \sum V(T^\xi, A)$, *where* $\sum$ *ranges over* $\xi \in \Omega_n^k$.

THEOREM (ii). *Let* $(T, A) \in \mathfrak{J}(k, n)$ *with* $H_n^{k+1}[T(A)] = 0$ *and* $H \subset A$ *be a set closed in* A *such that* $m_k[T^\xi(A^0 \cap H)] = 0$, $\xi \in \Omega_n^k$. *Then each component* A_i *of* $A - H$ *is an admissible set and* $V(T, A) = \sum V(T, A_i)$, $V(T^\xi, A) = \sum V(T^\xi, A_i)$, $\xi \in \Omega_n^k$, *where* $\sum$ *ranges over all components* A_i *of* $A - H$.

THEOREM (iii). *Let* $(T, A) \in \mathfrak{J}(k, n)$ *and* $H_n^{k+1}[T(A)] = 0$. *Let* $A_i \subset A$ $(i = 1, 2, \cdots)$ *be a collection of sets open in* A *such that each* A_i *is the union of sets* $g \in \Gamma(T, A)$. *Then we have* $V(T, \bigcup A_i) \leqq \sum V(T, A_i)$ *and* $V(T^\xi, \bigcup A_i) \leqq \sum V(T^\xi, A_i)$, $\xi \in \Omega_n^k$, *where* $\bigcup$ *and* $\sum$ *range over all* A_i.

The proofs of the above theorems are essentially the same as [3, (21.4.i), (21.5.i), (25.1.iii)].

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UNIVERSITY OF WISCONSIN, MILWAUKEE