DOUBLY ITERATED MATRIX METHODS OF SUMMABILITY[1]

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1. **Introduction.** A convenient generalization of the natural operator "limit" is realized in the concept of the summability-A of a sequence with respect to a matrix A. For $A=(a_{ik})$, the sequence $\{x_k\}$ is said to be summable-A to x provided

$$x = \lim_{i\to\infty} \lim_{j\to\infty} \sum_{k=0}^{j} a_{ik}x_k. \tag{1}$$

Here and throughout, elements of matrices and sequences are to be complex numbers. Indices run from 0 to ∞; and in ambiguous cases the sequence index will be repeated as final subscript in the fashion $\{c_{nk}\}_n$.

The matrix A is said to be *regular* if every convergent sequence is summable-A to its natural limit; for this the requirements on the a_{ik} are the celebrated Silverman-Toeplitz conditions [**1**, p. 64]. The idea of the present paper derives from the appeal to replace the application of the natural limit in (1) in both instances by summability-A itself, thereby yielding for regular A a not-less-general transform. More generally we consider the succession of functionals defined by repetitions of this double iteration.

Definition 1. With respect to a matrix $A=(a_{ik})$, the AI-Operator of order n, W_n, for $n=0, 1, 2, \cdots$, is the functional, operating on sequences, defined by the following recursion: $W_0\{x_k\}=\lim_{k\to\infty} x_k$, and $W_{n+1}\{x_k\}=W_n\{W_n\{\sum_{k=0}^{j} a_{ik}x_k\}_j\}_i$.

Thus $W_1\{x_k\}$ is the usual A-sum; and by way of example:

$$W_2\{x_k\} = \lim_{p\to\infty} \sum_{i=0}^{\infty} a_{pi} \lim_{m\to\infty} \sum_{j=0}^{\infty} a_{mj} \sum_{k=0}^{j} a_{ik}x_k.$$

Theorem 1. *Let A be a regular matrix for which W_m and W_n are the respective AI-Operators with $m \geq n$. Then for every sequence $\{c_k\}$ in the domain of definition of W_n, we have*

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$$W_m\{c_k\} = W_n\{c_k\}.$$

Proof. We consider only the case $m=n+1$; beyond that the proof is evident. For $n=0$, $m=1$, the conclusion is precisely the condition that A be regular, as hypothesized. Suppose the theorem is valid for $n=p$, $m=p+1$. To complete the induction, establishing the case $n=p+1$, $m=p+2$, we must, according to Definition 1, show that

$$W_{p+1}\left\{W_{p+1}\left\{\sum_{k=0}^{j} a_{ik}c_k\right\}_j\right\}_i = W_p\left\{W_p\left\{\sum_{k=0}^{j} a_{ik}c_k\right\}_j\right\}_i$$

whenever the expression on the right is defined. But from the assumption of the theorem's validity for $n=p$, $m=p+1$, we have for each i,

$$W_{p+1}\left\{\sum_{k=0}^{j} a_{ik}c_k\right\}_j = W_p\left\{\sum_{k=0}^{j} a_{ik}c_k\right\}_j$$

whenever the right side is defined. And similarly for the outside operators.

Definition 2. With respect to regular matrix A, a sequence $\{c_k\}$ is said to be summable-AI_ω to c, if for n sufficiently large we have $W_n\{c_k\}=c$.

Note. The relation $\{c_k\}$ summable-AI_ω to c may appropriately be written $W_\omega\{c_k\}=c$. Then as in Definition 1 we may define $W_{\omega+1}$; and so on to general ordinal number index. But the problems raised by such generality present a distraction from the classical application that follows and so the subject of transfinite indices is deferred.

In the next section we consider the application of the AI-Operators to sequences of functions, extending the idea and useful properties of uniform convergence. In §3 we generalize a result of conventional summability-A, establishing the effectiveness of the transforms W_n for summing Taylor Series in domains larger than the circle of convergence. Finally there is exhibited in §4 a simply derived matrix with respect to which the sequence of partial sums of each Taylor Series is summable-AI_ω to its analytic extension throughout the principal star domain of the function.

2. **Sequences of functions.** Definition 3. With respect to a matrix $A=(a_{ik})$, a sequence of functions $\{f_k(z)\}$ is said to be W_0-uniform for z in a set T if $\{f_k(z)\}$ is uniformly convergent for $z\in T$; and for $n=0, 1, 2, \cdots$, $\{f_k(z)\}$ is said to be W_{n+1}-uniform for $z\in T$ if for each i, $\{\sum_{k=0}^{j} a_{ik}f_k(z)\}_j$ is W_n-uniform for $z\in T$, and if in addition $\{W_n\{\sum_{k=0}^{j} a_{ik}f_k(z)\}_j\}_i$ is defined and W_n-uniform for $z\in T$.

It is immediate that "W_n-uniform" implies "W_n-summable." For conciseness we combine the concepts, writing simply: "$W_n\{f_k(z)\} = f(z)$ uniformly for $z \in T$." It is likewise clear that if $W_n\{c_k\}$ is defined, then considering $\{c_k\}$ as a sequence of functions constant over a set T it follows that $\{c_k\}$ is W_n-uniform for $z \in T$. Here, as in the theorems following, the matrix defining W_n is arbitrary, in particular it is not required to be regular.

THEOREM 2. *Suppose* $W_n\{f_k(z)\} = f(z)$ *and* $W_n\{g_k(z)\} = g(z)$, *both uniformly for* $z \in T$, *and let* $h(z)$ *be bounded for* $z \in T$. *Then*

$$W_n\{f_k(z) + h(z) \cdot g_k(z)\} = f(z) + h(z) \cdot g(z)$$

uniformly for $z \in T$.

THEOREM 3. *Suppose each element of* $\{f_k(z)\}$ *is continuous for* z *in a metric set* T; *suppose also that* $\{f_k(z)\}$ *is* W_n*-uniform for* $z \in T$. *Then* $W_n\{f_k(z)\}$ *is continuous for* $z \in T$.

THEOREM 4. *Suppose* $\{f_k(u, z)\}$ *is* W_n*-uniform for* (u, z) *in a set* $C \times T$ *where* C *is a rectifiable contour of the complex plane. Further suppose that for each* k *and each* $z \in T$, $f_k(u, z)$ *is continuous for* $u \in C$. *Then*

$$W_n\left\{\int_C f_k(u, z)du\right\} = \int_C W_n\{f_k(u, z)\}du$$

uniformly for $z \in T$.

The proofs of these three theorems all conform to the same induction format. In each of them the case $n = 0$ is commonplace, and the mechanics of passing from m to $m+1$ is straightforward. We illustrate with the details of the proof of Theorem 4.

PROOF OF THEOREM 4. As just observed the result for the case $n = 0$, ordinary uniform convergence, is well known. Assume the theorem valid for $n = m$. Let $\{f_k(u, z)\}$ satisfy the hypotheses for the case $n = m+1$. First observe that

$$\sum_{k=0}^{j} a_{ik} \int_C f_k(u, z)du = \int_C \sum_{k=0}^{j} a_{ik} f_k(u, z)du.$$

From the assumed W_{m+1}-uniformity of $\{f_k(u, z)\}$, $\{\sum_{k=0}^{j} a_{ik} f_k(u, z)\}_j$ is, for each i, W_m-uniform for $(u, z) \in C \times T$. The continuity of $\sum_{k=0}^{j} a_{ik} f_k(u, z)$ for $u \in C$ follows from the continuity of the respective $f_k(u, z)$. Therefore from the $n = m$ case of the theorem we have for each i:

$$W_m\left\{\int_C \sum_{k=0}^{j} a_{ik}f_k(u, z)du\right\}_j = \int_C W_m\left\{\sum_{k=0}^{j} a_{ik}f_k(u, z)\right\}_j du$$

uniformly for $z \in T$.

Again by assumption $\{W_m\{\sum_{k=0}^{j} a_{ik}f_k(u, z)\}_j\}_i$ is W_m-uniform for $(u, z) \in C \times T$. For each i and each $z \in T$ the continuity of $W_m\{\sum_{k=0}^{j} a_{ik}f_k(u, z)\}_j$ follows from Theorem 3. Thus again from the $n=m$ case of the theorem:

$$W_m\left\{\int_C W_m\left\{\sum_{k=0}^{j} a_{ik}f_k(u, z)\right\}_j du\right\}_i = \int_C W_m\left\{W_m\left\{\sum_{k=0}^{j} a_{ik}f_k(u, z)\right\}_j\right\}_i du$$

uniformly for $z \in T$.

Collecting the steps we have

$$W_m\left\{W_m\left\{\sum_{k=0}^{j} a_{ik}\int_C f_k(u, z)du\right\}_j\right\}_i = \int_C W_m\left\{W_m\left\{\sum_{k=0}^{j} a_{ik}f_k(u, z)\right\}_j\right\}_i du,$$

the sequences in j and i on the left being W_m-uniform for $z \in T$. This is precisely the desired $n=m+1$ result.

The linearity of the operators W_n we now observe as the all-functions-constant case of Theorem 2. The extension to the following statement is immediate.

THEOREM 5. *Each AI-Operator W_n, as well as the summability-AI_ω transform for regular A, defines a linear functional over a vector space of sequences of complex numbers.*

3. **Application to Taylor series.** Applying an AI-Operator W_n to the partial sums of the Geometric Series, we have:

$$W_n\left\{\sum_{k=0}^{j} z^k\right\}_j = W_n\left\{\frac{1-z^{j+1}}{1-z}\right\}_j = \frac{1}{1-z}\left[W_n\{1\}_j - zW_n\{z^j\}_j\right]$$

provided the expression on the right exists. ($\{1\}_j$ represents the sequence of all 1's). Thus sufficient conditions that W_n "properly" sum the Geometric Series at a point z are that $W_n\{1\}_j=1$ and $W_n\{z^j\}_j=0$. Theorem 6 provides an analogous result for Taylor Series in general.

A function of the form $f(z)=\sum_{k=0}^{\infty} c_kz^k$ with positive radius of convergence will be regarded as extended to its *principal star domain*, i.e., if there exists an analytic continuation of $f(z)$ throughout a domain containing the segment $\{tz_0 | 0 \leqq t \leqq 1\}$ then $f(z_0)$ represents the value defined thereby. We represent the principal star domain as M_f. Note that its complement $\mathcal{C}M_f$ consists of those points which are singularities of the analytic function $f(z)$ by a radial approach, automatically including all points "in the shadow of" such singularities.

In general a region Q is said to be *starlike* if for each $z \in Q$ we have $tz \in Q$ for $0 \leqq t \leqq 1$. For a starlike domain Q, the *partial star domain* P_{fQ} of $f(z)$ with respect to Q is the intersection of the sets ζQ as ζ ranges over $\mathcal{C}M_f$. (ζQ represents the set $\{\zeta z | z \in Q\}$).

The following result is an extension of a theorem of Okada [1, p. 189] applying for conventional matrix summability.

THEOREM 6. *Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ have positive radius of convergence, and let $s_j(z) = \sum_{k=0}^{j} c_k z^k$. Let W_n be an AI-Operator with the properties that $W_n\{1\}_k = 1$ and $W_n\{z^k\}_k = 0$ uniformly for z in each closed and bounded set in a starlike domain Q. Then $W_n\{s_j(z)\} = f(z)$ uniformly for z in each closed and bounded set in the partial star domain P_{fQ} of $f(z)$ with respect to Q.*

PROOF. Since Q clearly cannot contain the point $z=1$ it follows that P_{fQ} is a subset of M_f. And since otherwise the theorem is vacuous we assume that Q contains the origin.

Let T represent a closed and bounded set in P_{fQ}. For $j = 0, 1, 2, \cdots$, consider the integral:

$$I_j(z) = \frac{1}{2\pi i} \int_C \frac{f(u)}{z - u} \left(\frac{z}{u}\right)^{j+1} du. \tag{2}$$

Here C is a rectifiable simple closed curve, taken counterclockwise, with the properties: $f(z)$ is analytic on and inside C; the origin and the set T are properly inside C; and the union of all points of the form t/u for $t \in T$ and $u \in C$ forms a closed and bounded subset of Q.

To verify the existence of such a contour C without a tedious direct construction we note that since M_f clearly satisfies the hypotheses of the Riemann Mapping Theorem there is an analytic function $g(z)$ which simply maps the unit circle $|z| < 1$ onto M_f. Since T is a closed and bounded set in M_f the pre-image of T under this mapping will be contained in a circle $|z| < \rho_1 < 1$. Furthermore the definition of P_{fQ} insures that for all $t \in T \subset P_{fQ}$ and $\zeta \in \mathcal{C}M_f$ we have $(t/\zeta) \in Q$. Thus if $\gamma \in \mathcal{C}Q$ it follows that $(t/\gamma) \in M_f$. Let S represent the union of the origin and all points of the form t/γ as t ranges over the closed and bounded set T and γ ranges over the closed and bounded-away-from-zero set $\mathcal{C}Q$. Clearly S is a closed and bounded set in M_f. As with T, the image of S under the inverse mapping $g^{-1}(z)$ lies in a circle $|z|<\rho_2<1$. To recapitulate: $t\in T$ implies $|g^{-1}(t)|<\rho_1<1$; and $t\in T$ and $\rho_2\leq|z|<1$ implies $(t/g(z))\in Q$. It follows immediately that the image under $g(z)$ of the circle $|z|=\max(\rho_1, \rho_2)$ furnishes an acceptable contour C.

The integrand in (2) is analytic inside C except for poles at $u=0$ and $u=z$. Observing that $I_j(0)=0$ and $f(0)=s_j(0)=c_0$, we pass on to the case $z\neq 0$. The residue at $u=z$ is clearly $-f(z)$. Near $u=0$ the integrand may be expanded thus:

$$\frac{z^{j+1}}{u^{j+1}}\cdot\frac{1}{z}(c_0+c_1u+\cdots+c_ku^k+\cdots)\cdot\left(1+\frac{u}{z}+\cdots+\frac{u^k}{z^k}+\cdots\right).$$

Collecting the coefficient of u^{-1} we have the residue:

$$z^j\left(\frac{c_0}{z^j}+\frac{c_1}{z^{j-1}}+\cdots+c_j\right)=s_j(z).$$

It follows therefore that for all $z\in T$,

$$I_j(z)=s_j(z)-f(z).$$

From the hypotheses of the theorem, Theorems 2 and 4, and the properties of C, it is clear that

$$W_n\{I_j(z)\}_j=\frac{1}{2\pi i}\int_C\frac{f(u)}{z-u}\cdot\frac{z}{u}\cdot W_n\left\{\left(\frac{z}{u}\right)^j\right\}_j du=0$$

uniformly for $z\in T$. Recalling Theorem 2 and the fact that $W_n\{1\}_k=1$, we have finally

$$W_n\{I_j(z)+f(z)\}_j=f(z)$$

uniformly for $z\in T$. Since

$$I_j(z)+f(z)=s_j(z),$$

this is the desired result.

4. **An example.** Henceforth let $G(z)=(2-z)^{-1}$, and let $A=(a_{ik})$ be the matrix whose elements are the coefficients of the Taylor Series of the functions $[G(z)]^i$. To be precise

$$[G(z)]^i=(2-z)^{-i}=\sum_{k=0}^{\infty}a_{ik}z^k \qquad \text{for } |z|<2. \tag{3}$$

Hence

$$a_{ik} = (1/2)^{i+k} C_{i+k-1,i-1}. \tag{4}$$

This matrix A, or more specifically the summability-A functional associated with it, represents the $\alpha = 1/2$ case of the S_α methods which correspond to the family of matrices defined in the manner of (3) from the functions $G_\alpha(z) = (1-\alpha)/(1-\alpha z)$ for $0 < \alpha < 1$ [2]. That A is regular follows immediately from the Silverman-Toeplitz Conditions. However we note that it is an easy exercise to demonstrate by an (ϵ, N)-type argument that a given convergent sequence is summable-A to its natural limit. The rest of the paper is devoted to demonstrating the following result.

THEOREM 7. *Let W_n be the AI-Operator of order n for the matrix A given by* (4). *Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ have positive radius of convergence, and let $s_j(z) = \sum_{k=0}^{j} c_k z^k$. Then for each closed and bounded set T in the principal star domain M_f of $f(z)$, there is an N such that for all $n > N$,*

$$W_n\{s_j(z)\}_j = f(z)$$

uniformly for $z \in T$.

For a set T consisting of a single point, Theorem 7 takes the form:

COROLLARY. *Let A, $f(z)$ and $s_j(z)$ be as in Theorem 7. Then $\{s_j(z)\}$ is summable-AI_ω to $f(z)$ throughout the principal star domain of $f(z)$.*

In point of fact there have been exhibited matrices with respect to which the usual summability method (our W_1) sums $\{s_j(z)\}$ to $f(z)$ throughout M_f [1, pp. 181–187]. But the construction of our matrix is very much simpler, and although summability-AI_ω arithmetically transcends ordinary summability-A it is nonetheless a perfectly natural generalization. Moreover the proof of Theorem 7 requires no more that an application on Theorem 6 together with a straightforward geometrical argument. The argument may clearly be adapted to a general class of matrices; in particular we note that Theorem 7 may be established in similar fashion for the matrices of all S_α methods where $1/2 \leqq \alpha < 1$.

PROOF OF THEOREM 7. Since the matrix A is regular it follows from Theorem 1 that $W_n\{1\}_k = 1$ for all n. Suppose for given n, $W_n\{z^k\}_k = 0$ uniformly in each closed set in a bounded starlike domain R_n. Then applying Theorem 6 it follows that $W_n\{\sum_{k=0}^{j} a_{ik} z^k\}_j = [G(z)]^i$ for $z \in 2R_n$, and $W_n\{[G(z)]^i\}_i = 0$ for $z \in G^{-1}(R_n)$, both with the usual uniformity in closed subsets. (Here, $2R_n = \{z \mid (z/2) \in R_n\}$ and $G^{-1}(R_n) = \{z \mid G(z) \in R_n\}$.) Thus if R_{n+1} is a bounded starlike domain contained in $G^{-1}(R_n) \cap 2R_n$ then $W_{n+1}\{z^k\}_k = 0$ uniformly in each closed set in R_{n+1}.

With $G^0(z)=z$ and $G^{n+1}(z)=G(G^n(z))$, let D_n represent the domain in which $|G^n(z)|<1$. Thus D_0 is the unit circle; and with the usual linear fractional transformation manipulations it follows inductively that for $n=1, 2, \cdots, D_n$ is the domain defined by the following inequality:

$$D_n: \left| z - \left(1 + \frac{1}{2n-1}\right)\right| > \frac{1}{2n-1} \cdot \tag{5}$$

Let D_n^* represent the starlike part of D_n, i.e., D_n^* consists of those points z of D_n for which the segment $\{tz \mid 0 \leqq t \leqq 1\}$ does not intersect the circle bounding D_n. Finally let

$$Q_n = \bigcap_{k=0}^{n} 2^k D_{n-k}^*. \tag{6}$$

From the definition it follows that Q_n is bounded and starlike, that $Q_{n+1}=D_{n+1}^* \cap 2Q_n$ and that $Q_{n+1} \supset Q_n$ with $\bigcup_{n=0}^{\infty} Q_n$ being precisely the complement of the segment $\{x \mid 1 \leqq x\}$.

Lemma. *With W_n defined as in Theorem 7, $W_n\{z^k\}_k=0$ uniformly for z in each closed set in the domain Q_n of* (6).

Proof of Lemma. Since Q_0 is the unit circle, the $n=0$ (natural convergence) case of the lemma is immediate. To deduce the validity of the lemma for $n+1$ assuming its validity for n, we employ the argument spelled out above in terms of domains R_n and R_{n+1}. That Q_n is bounded and starlike has already been observed, so we have only to demonstrare that $Q_{n+1} \subset G^{-1}(Q_n) \cap 2Q_n$. But as has also been observed, $Q_{n+1}=D_{n+1}^* \cap 2Q_n$, so it suffices to show that $D_{n+1}^* \subset G^{-1}(Q_n)$. Referring to (6) the proof reduces to showing that $D_{n+1}^* \subset G^{-1}(2^k D_{n-k}^*)$ for $k=0, 1, \cdots, n$.

The character of the domains D_{n+1}^*, $2^k D_{n-k}^*$ and $G^{-1}(2^k D_{n-k}^*)$, for $k=0, 1, \cdots, n-1$, is indicated in Figure 1. The former two are plotted directly from definition; the construction of $G^{-1}(2^k D_{n-k}^*)$ for the transformation $G^{-1}(z)=2-z^{-1}$ is immediate. From the geometry it is clear that the condition $D_{n+1}^* \subset G^{-1}(2^k D_{n-k}^*)$ is equivalent to the following inequality comparing the ratio of diameter to distance from the origin for the circles generating D_{n+1}^* and $G^{-1}(2^k D_{n-k}^*)$ respectively:

$$\begin{aligned} \frac{2}{2n+1} \geqq \frac{1}{2-2^{-k}} \Bigg[\left(2 - \left(2^k\left(1 + \frac{2}{2(n-k)-1}\right)\right)^{-1}\right) \\ - (2-2^{-k})\Bigg]. \end{aligned} \tag{7}$$

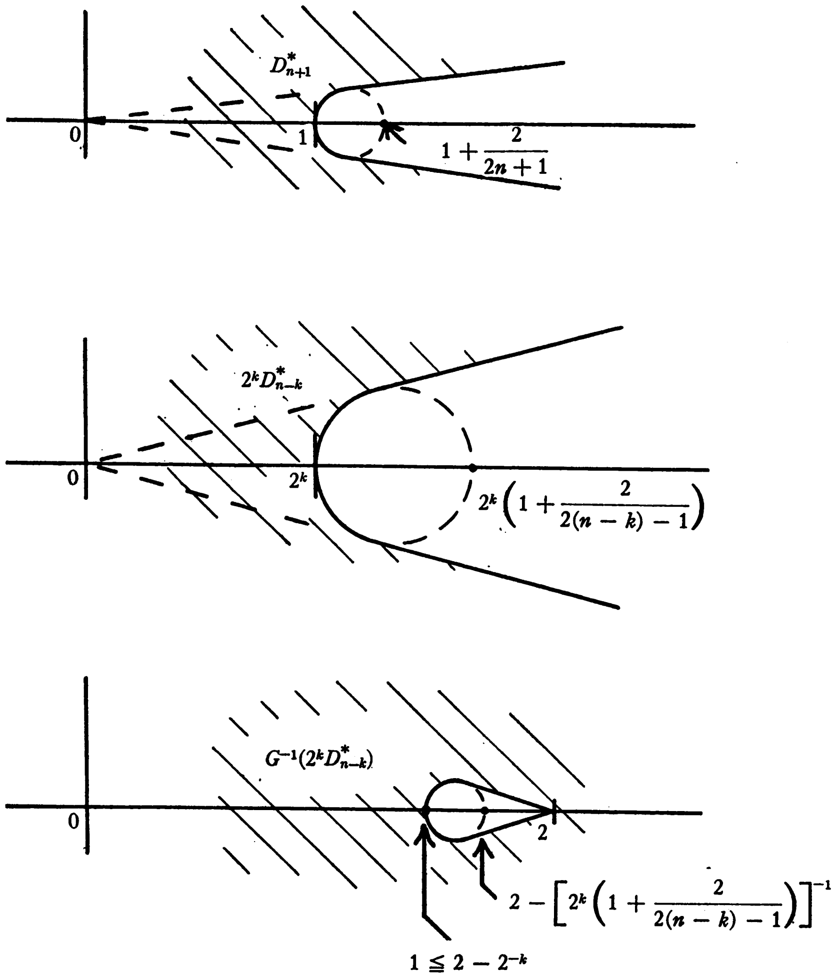

FIGURE 1

In the exceptional case $k=n$, the region $2^k D_{n-k}^*$ is simply the circle $|z|<2^n$, and $G^{-1}(2^k D)_{n-k}^*$ is the domain $|z-2|>2^{-n}$. Reasoning as before we find that the condition $D_{n+1}^* \subset G^{-1}(2^k D_{n-k}^*)$ for $k=n$ is indeed equivalent to the $k=n$ case of the inequality (7). Thus the proof of the lemma has reduced to proving the inequality (7) for $n=0, 1, 2, \cdots$ with $k=0, 1, \cdots, n$. Simplifying (7), we obtain the following equivalent forms:

$$\frac{2}{2n+1} \geqq \frac{1}{2^{k+1}-1}\left[1 - \frac{2(n-k)-1}{2(n-k)+1}\right] = \frac{2}{(2^{k+1}-1)(2(n-k)+1)},$$

$$(2^{k+1}-1)(2(n-k)+1) \geqq 2n+1,$$

$$(2^{k+1}-2)\cdot 2(n-k) + (2^{k+1}-2k-2) \geqq 0.$$

Since in the last form both terms on the left are clearly non-negative for admissible values of n and k, the desired inequality is established. This completes the proof of the lemma.

Returning to the proof of the theorem proper, recall that for the domains Q_n of (6), $Q_{n+1} \supset Q_n$ and $\bigcup_{n=0}^{\infty} Q_n$ omits only the half-line $[1, \infty]$. Since the principal star domain M_f is starlike, $z \in M_f$ and $\zeta \in \mathcal{C}M_f$ implies $(z/\zeta) \notin [1, \infty]$. Thus for a closed and bounded set T contained in M_f, the union of the origin and all points of the form z/ζ for $z \in T$ and $\zeta \in \mathcal{C}M_f$ forms a closed and bounded set V not intersecting $[1, \infty]$. Clearly for n sufficiently large, $V \subset Q_n$. But for $z \in T$: $\zeta \in \mathcal{C}M_f$ implies $(z/\zeta) \in V \subset Q_n$, or in other words, $z \in \zeta Q_n$; and thus $z \in P_{fQ_n}$. Therefore T is a closed and bounded set in P_{fQ_n}. Recalling the lemma and the fact that $W_n\{1\}_k = 1$, and applying Theorem 6, the conclusion of Theorem 7 follows.

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