Doubly Iterated Matrix Methods of Summability\textsuperscript{1}

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1. Introduction. A convenient generalization of the natural operator "limit" is realized in the concept of the summability-$A$ of a sequence with respect to a matrix $A$. For $A = (a_{ik})$, the sequence $\{x_k\}$ is said to be summable-$A$ to $x$ provided

\begin{equation}
\lim_{t \to \infty} \lim_{j \to \infty} \sum_{k=0}^{j} a_{ik} x_k.
\end{equation}

Here and throughout, elements of matrices and sequences are to be complex numbers. Indices run from 0 to $\infty$; and in ambiguous cases the sequence index will be repeated as final subscript in the fashion $\{x_k\}_{n}$.

The matrix $A$ is said to be regular if every convergent sequence is summable-$A$ to its natural limit; for this the requirements on the $a_{ik}$ are the celebrated Silverman-Toeplitz conditions [1, p. 64]. The idea of the present paper derives from the appeal to replace the application of the natural limit in (1) in both instances by summability-$A$ itself, thereby yielding for regular $A$ a not-less-general transform. More generally we consider the succession of functionals defined by repetitions of this double iteration.

Definition 1. With respect to a matrix $A = (a_{ik})$, the $AI$-Operator of order $n$, $W_n$, for $n = 0, 1, 2, \ldots$, is the functional, operating on sequences, defined by the following recursion: $W_0 \{x_k\} = \lim_{k \to \infty} x_k$, and $W_{n+1} \{x_k\} = W_n \{ W_n \{ \sum_{k=0}^{n} a_{ik} x_k \} \}$.

Thus $P_{F1}\{x_k\}$ is the usual $A$-sum; and by way of example:

\begin{equation}
W_2 \{x_k\} = \lim_{p \to \infty} \sum_{i=0}^{\infty} a_{pi} \lim_{m \to \infty} \sum_{j=0}^{\infty} a_{mj} \sum_{k=0}^{j} a_{ik} x_k.
\end{equation}

Theorem 1. Let $A$ be a regular matrix for which $W_m$ and $W_n$ are the respective $AI$-Operators with $m \geq n$. Then for every sequence $\{c_k\}$ in the domain of definition of $W_n$, we have

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Proof. We consider only the case \( m = n + 1 \); beyond that the proof is evident. For \( n = 0, m = 1 \), the conclusion is precisely the condition that \( A \) be regular, as hypothesized. Suppose the theorem is valid for \( n = p, m = p + 1 \). To complete the induction, establishing the case \( n = p + 1, m = p + 2 \), we must, according to Definition 1, show that

\[
W_{p+1}\left\{ \sum_{k=0}^{j} a_{ik} c_k \right\}_i = W_p W_{p+1} \left\{ \sum_{k=0}^{j} a_{ik} c_k \right\}_i
\]

whenever the expression on the right is defined. But from the assumption of the theorem's validity for \( n = p, m = p + 1 \), we have for each \( i \),

\[
W_{p+1}\left\{ \sum_{k=0}^{j} a_{ik} c_k \right\}_j = W_p \left\{ \sum_{k=0}^{j} a_{ik} c_k \right\}_j
\]

whenever the right side is defined. And similarly for the outside operators.

Definition 2. With respect to regular matrix \( A \), a sequence \( \{ c_k \} \) is said to be summable-\( A I_n \) to \( c \), if for \( n \) sufficiently large we have \( W_n\{ c_k \} = c \).

Note. The relation \( \{ c_k \} \) summable-\( A I_n \) to \( c \) may appropriately be written \( W_n\{ c_k \} = c \). Then as in Definition 1 we may define \( W_{n+1} \); and so on to general ordinal number index. But the problems raised by such generality present a distraction from the classical application that follows and so the subject of transfinite indices is deferred.

In the next section we consider the application of the \( A I \)-Operators to sequences of functions, extending the idea and useful properties of uniform convergence. In §3 we generalize a result of conventional summability-\( A \), establishing the effectiveness of the transforms \( W_n \) for summing Taylor Series in domains larger than the circle of convergence. Finally there is exhibited in §4 a simply derived matrix with respect to which the sequence of partial sums of each Taylor Series is summable-\( A I_n \) to its analytic extension throughout the principal star domain of the function.

2. Sequences of functions. Definition 3. With respect to a matrix \( A = (a_{ik}) \), a sequence of functions \( \{ f_k(z) \} \) is said to be \( W_n\)-uniform for \( z \) in a set \( T \) if \( \{ f_k(z) \} \) is uniformly convergent for \( z \in T \); and for \( n = 0, 1, 2, \ldots \), \( \{ f_k(z) \} \) is said to be \( W_{n+1}\)-uniform for \( z \in T \) if for each \( i \), \( \left\{ \sum_{k=0}^{j} a_{ik} f_k(z) \right\}_i \) is \( W_n\)-uniform for \( z \in T \), and if in addition \( \left\{ W_n \left\{ \sum_{k=0}^{j} a_{ik} f_k(z) \right\}_i \right\}_i \) is defined and \( W_n\)-uniform for \( z \in T \).
It is immediate that "$W_n$-uniform" implies "$W_n$-summable." For conciseness we combine the concepts, writing simply: "$W_n\{f_k(z)\} = f(z)$ uniformly for $z \in T."$ It is likewise clear that if $W_n\{c_k\}$ is defined, then considering $\{c_k\}$ as a sequence of functions constant over a set $T$ it follows that $\{c_k\}$ is $W_n$-uniform for $z \in T$. Here, as in the theorems following, the matrix defining $W_n$ is arbitrary, in particular it is not required to be regular.

**Theorem 2.** Suppose $W_n\{f_k(z)\} = f(z)$ and $W_n\{g_k(z)\} = g(z)$, both uniformly for $z \in T$, and let $h(z)$ be bounded for $z \in T$. Then

$$W_n\{f_k(z) + h(z) \cdot g_k(z)\} = f(z) + h(z) \cdot g(z)$$

uniformly for $z \in T$.

**Theorem 3.** Suppose each element of $\{f_k(z)\}$ is continuous for $z$ in a metric set $T$; suppose also that $\{f_k(z)\}$ is $W_n$-uniform for $z \in T$. Then $W_n\{f_k(z)\}$ is continuous for $z \in T$.

**Theorem 4.** Suppose $\{f_k(u, z)\}$ is $W_n$-uniform for $(u, z)$ in a set $C \times T$ where $C$ is a rectifiable contour of the complex plane. Further suppose that for each $k$ and each $z \in T$, $f_k(u, z)$ is continuous for $u \in C$. Then

$$W_n\left\{ \int_C f_k(u, z) \, du \right\} = \int_C W_n\{f_k(u, z)\} \, du$$

uniformly for $z \in T$.

The proofs of these three theorems all conform to the same induction format. In each of them the case $n = 0$ is commonplace, and the mechanics of passing from $m$ to $m + 1$ is straightforward. We illustrate with the details of the proof of Theorem 4.

**Proof of Theorem 4.** As just observed the result for the case $n = 0$, ordinary uniform convergence, is well known. Assume the theorem valid for $n = m$. Let $\{f_k(u, z)\}$ satisfy the hypotheses for the case $n = m + 1$. First observe that

$$\sum_{k=0}^{j} a_{ik} \int_C f_k(u, z) \, du = \int_C \sum_{k=0}^{j} a_{ik} f_k(u, z) \, du.$$

From the assumed $W_{m+1}$-uniformity of $\{f_k(u, z)\}$, $\{\sum_{k=0}^{j} a_{ik} f_k(u, z)\}$ is, for each $i$, $W_m$-uniform for $(u, z) \in C \times T$. The continuity of $\sum_{k=0}^{j} a_{ik} f_k(u, z)$ for $u \in C$ follows from the continuity of the respective $f_k(u, z)$. Therefore from the $n = m$ case of the theorem we have for each $i$: 
\[ W_m \left\{ \int_C \sum_{k=0}^j a_{ik} f_k(u, z) \, du \right\}_j = \int_C W_m \left\{ \sum_{k=0}^j a_{ik} f_k(u, z) \right\}_j \, du \]

uniformly for \( z \in T \).

Again by assumption \( \{ W_m \left\{ \sum_{k=0}^i a_{ik} f_k(u, z) \right\}_i \} \) is \( W_m \)-uniform for \( (u, z) \in C \times T \). For each \( i \) and each \( z \in T \) the continuity of \( W_m \left\{ \sum_{k=0}^i a_{ik} f_k(u, z) \right\}_i \) follows from Theorem 3. Thus again from the \( n=m \) case of the theorem:

\[ W_m \left\{ \int_C W_m \left\{ \sum_{k=0}^i a_{ik} f_k(u, z) \right\}_i \, du \right\}_j = \int_C W_m \left\{ \sum_{k=0}^i a_{ik} f_k(u, z) \right\}_i \, du \]

uniformly for \( z \in T \).

Collecting the steps we have

\[ W_m \left\{ W_m \left\{ \sum_{k=0}^j a_{ik} \int_C f_k(u, z) \, du \right\}_j \right\}_i = \int_C W_m \left\{ \sum_{k=0}^j a_{ik} f_k(u, z) \right\}_j \, du, \]

the sequences in \( j \) and \( i \) on the left being \( W_m \)-uniform for \( z \in T \). This is precisely the desired \( n=m+1 \) result.

The linearity of the operators \( W_n \) we now observe as the all-functions-constant case of Theorem 2. The extension to the following statement is immediate.

**Theorem 5.** Each AL-Operator \( W_n \), as well as the summability-AL\(_n\) transform for regular \( A \), defines a linear functional over a vector space of sequences of complex numbers.

3. Application to Taylor series. Applying an AL-Operator \( W_n \) to the partial sums of the Geometric Series, we have:

\[ W_n \left\{ \sum_{k=0}^j z^k \right\}_j = W_n \left\{ \frac{1 - z^{i+1}}{1 - z} \right\}_j = \frac{1}{1 - z} [W_n \{1\}_j - zW_n \{z^i\}_j] \]

provided the expression on the right exists. (\( \{1\}_j \) represents the sequence of all 1's). Thus sufficient conditions that \( W_n \) "properly" sum the Geometric Series at a point \( z \) are that \( W_n \{1\}_j = 1 \) and \( W_n \{z^i\}_j = 0 \). Theorem 6 provides an analogous result for Taylor Series in general.

A function of the form \( f(z) = \sum_{k=0}^\infty c_k z^k \) with positive radius of convergence will be regarded as extended to its principal star domain,
i.e., if there exists an analytic continuation of \( f(z) \) throughout a domain containing the segment \( \{ tz_0 | 0 \leq t \leq 1 \} \) then \( f(z_0) \) represents the value defined thereby. We represent the principal star domain as \( M_f \).

Note that its complement \( \partial M_f \) consists of those points which are singularities of the analytic function \( f(z) \) by a radial approach, automatically including all points “in the shadow of” such singularities.

In general a region \( Q \) is said to be starlike if for each \( z \in Q \) we have \( tz \in Q \) for \( 0 \leq t \leq 1 \). For a starlike domain \( Q \), the partial star domain \( P_{fQ} \) of \( f(z) \) with respect to \( Q \) is the intersection of the sets \( \{ \xi \} \), \( \xi \in Q \).

The following result is an extension of a theorem of Okada [1, p. 189] applying for conventional matrix summability.

**Theorem 6.** Let \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) have positive radius of convergence, and let \( s_j(z) = \sum_{k=0}^{\infty} c_k z^k \). Let \( W_n \) be an \( AI \)-Operator with the properties that \( W_n \{ 1 \} = 1 \) and \( W_n \{ z^k \} = 0 \) uniformly for \( z \) in each closed and bounded set in a starlike domain \( Q \). Then \( W_n \{ s_j(z) \} = f(z) \) uniformly for \( z \) in each closed and bounded set in the partial star domain \( P_{fQ} \) with respect to \( Q \).

**Proof.** Since \( Q \) clearly cannot contain the point \( z = 1 \) it follows that \( P_{fQ} \) is a subset of \( M_f \). And since otherwise the theorem is vacuous we assume that \( Q \) contains the origin.

Let \( T \) represent a closed and bounded set in \( P_{fQ} \). For \( j = 0, 1, 2, \ldots \), consider the integral:

\[
I_j(z) = \frac{1}{2\pi i} \int_C \frac{f(u)}{z - u} \left( \frac{z}{u} \right)^{j+1} du.
\]

Here \( C \) is a rectifiable simple closed curve, taken counterclockwise, with the properties: \( f(z) \) is analytic on and inside \( C \); the origin and the set \( T \) are properly inside \( C \); and the union of all points of the form \( t/u \) for \( t \in T \) and \( u \in C \) forms a closed and bounded subset of \( Q \).

To verify the existence of such a contour \( C \) without a tedious direct construction we note that since \( M_f \) clearly satisfies the hypotheses of the Riemann Mapping Theorem there is an analytic function \( g(z) \) which simply maps the unit circle \( |z| < 1 \) onto \( M_f \). Since \( T \) is a closed and bounded set in \( M_f \) the pre-image of \( T \) under this mapping will be contained in a circle \( |z| < \rho \), \( \rho < 1 \). Furthermore the definition of \( P_{fQ} \) insures that for all \( t \in T \subseteq P_{fQ} \) and \( \xi \in \partial M_f \) we have \( (t/\xi) \subseteq Q \).

Thus if \( \gamma \in \partial Q \) it follows that \( (t/\gamma) \subseteq M_f \). Let \( S \) represent the union of the origin and all points of the form \( t/\gamma \) as \( t \) ranges over the closed and bounded set \( T \) and \( \gamma \) ranges over the closed and bounded-away-from-zero set \( \partial M_f \). Clearly \( S \) is a closed and bounded set in \( M_f \). As with \( T \),
the image of $S$ under the inverse mapping $g^{-1}(z)$ lies in a circle $|z| < r_2 < 1$. To recapitulate: $t \in T$ implies $|g^{-1}(t)| < r_1 < 1$; and $t \in T$ and $r_2 \leq |z| < 1$ implies $(t/g(z)) \in Q$. It follows immediately that the image under $g(z)$ of the circle $|z| = \max(r_1, r_2)$ furnishes an acceptable contour $C$.

The integrand in (2) is analytic inside $C$ except for poles at $u=0$ and $u=z$. Observing that $I_j(0) = 0$ and $f(0) = s_j(0) = c_0$, we pass on to the case $z \neq 0$. The residue at $u = z$ is clearly $-f(z)$. Near $u = 0$ the integrand may be expanded thus:

$$
\frac{z^{i+1}}{u^{i+1}} \cdot \frac{1}{z} \left( c_0 + c_1 u + \cdots + c_k u^k + \cdots \right) \cdot \left( 1 + \frac{u}{z} + \cdots + \frac{u^k}{z^k} + \cdots \right).
$$

Collecting the coefficient of $u^{-1}$ we have the residue:

$$
z^i \left( \frac{c_0}{z^i} + \frac{c_1}{z^{i-1}} + \cdots + c_i \right) = s_j(z).
$$

It follows therefore that for all $z \in T$,

$$
I_j(z) = s_j(z) - f(z).
$$

From the hypotheses of the theorem, Theorems 2 and 4, and the properties of $C$, it is clear that

$$
W_n\{I_j(z)\}_{j} = \frac{1}{2\pi i} \int_{C} \frac{f(u)}{z-u} \cdot \cdot \cdot W_n\{(Z)_j\} \cdot du = 0
$$

uniformly for $z \in T$. Recalling Theorem 2 and the fact that $W_n\{1\}_{k} = 1$, we have finally

$$
W_n\{I_j(z) + f(z)\}_{j} = f(z)
$$

uniformly for $z \in T$. Since

$$
I_j(z) + f(z) = s_j(z),
$$

this is the desired result.

4. An example. Henceforth let $G(z) = (2-z)^{-1}$, and let $A = (a_{ik})$ be the matrix whose elements are the coefficients of the Taylor Series of the functions $[G(z)]^i$. To be precise

$$
[G(z)]^i = (2-z)^{-i} = \sum_{k=0}^{\infty} a_{ik} z^k \quad \text{for } |z| < 2.
$$

Hence
This matrix $A$, or more specifically the summability-$A$ functional associated with it, represents the $\alpha = 1/2$ case of the $S_\alpha$ methods which correspond to the family of matrices defined in the manner of (3) from the functions $G_\alpha(z) = (1 - \alpha)/(1 - \alpha z)$ for $0 < \alpha < 1$ [2]. That $A$ is regular follows immediately from the Silverman-Toeplitz Conditions. However we note that it is an easy exercise to demonstrate by an $(\epsilon, N)$-type argument that a given convergent sequence is summable-$A$ to its natural limit. The rest of the paper is devoted to demonstrating the following result.

**Theorem 7.** Let $W_n$ be the AI-Operator of order $n$ for the matrix $A$ given by (4). Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ have positive radius of convergence, and let $s_j(z) = \sum_{k=0}^{\infty} d_k z^k$. Then for each closed and bounded set $T$ in the principal star domain $M_f$ of $f(z)$, there is an $N$ such that for all $n > N$,

$$W_n\{s_j(z)\}_{j=1} = f(z)$$

uniformly for $z \in T$.

For a set $T$ consisting of a single point, Theorem 7 takes the form:

**Corollary.** Let $A$, $f(z)$ and $s_j(z)$ be as in Theorem 7. Then $\{s_j(z)\}$ is summable-$AI_a$ to $f(z)$ throughout the principal star domain of $f(z)$.

In point of fact there have been exhibited matrices with respect to which the usual summability method (our $W_1$) sums $\{s_j(z)\}$ to $f(z)$ throughout $M_f$ [1, pp. 181-187]. But the construction of our matrix is very much simpler, and although summability-$AI_a$ arithmetically transcends ordinary summability-$A$ it is nonetheless a perfectly natural generalization. Moreover the proof of Theorem 7 requires no more that an application on Theorem 6 together with a straightforward geometrical argument. The argument may clearly be adapted to a general class of matrices; in particular we note that Theorem 7 may be established in similar fashion for the matrices of all $S_\alpha$ methods where $1/2 \leq \alpha < 1$.

**Proof of Theorem 7.** Since the matrix $A$ is regular it follows from Theorem 1 that $W_n\{1\}_{k=1} = 1$ for all $n$. Suppose for given $n$, $W_n\{z^k\}_{k=1} = 0$ uniformly in each closed set in a bounded starlike domain $R_n$. Then applying Theorem 6 it follows that $W_n\{\sum_{k=0}^{\infty} a_{ik} z^k\}_{j=1} = [G(z)]^i$ for $z \in 2R_n$, and $W_n\{[G(z)]^i\}_{i=1} = 0$ for $z \in G^{-1}(R_n)$, both with the usual uniformity in closed subsets. (Here, $2R_n = \{z| (z/2) \in R_n\}$ and $G^{-1}(R_n) = \{z| G(z) \in R_n\}$.) Thus if $R_{n+1}$ is a bounded starlike domain contained in $G^{-1}(R_n) \cap 2R_n$ then $W_{n+1}\{z^k\}_{k=1} = 0$ uniformly in each closed set in $R_{n+1}$. 

With \( G^0(z) = z \) and \( G^{n+1}(z) = G(G^n(z)) \), let \( D_n \) represent the domain in which \( |G^n(z)| < 1 \). Thus \( D_0 \) is the unit circle; and with the usual linear fractional transformation manipulations it follows inductively that for \( n = 1, 2, \ldots, D_n \) is the domain defined by the following inequality:

\[
D_n: \left| z - \left( 1 + \frac{1}{2n - 1} \right) \right| > \frac{1}{2n - 1}.
\]

Let \( D_n^* \) represent the starlike part of \( D_n \), i.e., \( D_n^* \) consists of those points \( z \) of \( D_n \) for which the segment \( \{ tz : 0 \leq t \leq 1 \} \) does not intersect the circle bounding \( D_n \). Finally let

\[
Q_n = \bigcap_{k=0}^{n} 2^k D_{n-k}^*.
\]

From the definition it follows that \( Q_n \) is bounded and starlike, that \( Q_{n+1} = D_{n+1}^* \cap 2Q_n \) and that \( Q_{n+1} \supseteq Q_n \) with \( \bigcup_{n=0}^{\infty} Q_n \) being precisely the complement of the segment \( \{ x : 1 \leq x \} \).

**Lemma.** With \( W_n \) defined as in Theorem 7, \( W_n \{ z^k \}_{k=0}^{n} = 0 \) uniformly for \( z \) in each closed set in the domain \( Q_n \) of (6).

**Proof of Lemma.** Since \( Q_0 \) is the unit circle, the \( n = 0 \) (natural convergence) case of the lemma is immediate. To deduce the validity of the lemma for \( n+1 \) assuming its validity for \( n \), we employ the argument spelled out above in terms of domains \( R_n \) and \( R_{n+1} \). That \( Q_n \) is bounded and starlike has already been observed, so we have only to demonstrate that \( Q_{n+1} \subset G^{-1}(Q_n) \cap 2Q_n \). But as has also been observed, \( Q_{n+1} = D_{n+1}^* \cap 2Q_n \), so it suffices to show that \( D_{n+1}^* \subset G^{-1}(Q_n) \). Referring to (6) the proof reduces to showing that \( D_{n+1}^* \subset G^{-1}(2^k D_{n-k}^*) \) for \( k = 0, 1, \ldots, n \).

The character of the domains \( D_{n+1}^*, 2^k D_{n-k}^* \) and \( G^{-1}(2^k D_{n-k}^*) \), for \( k = 0, 1, \ldots, n-1 \), is indicated in Figure 1. The former two are plotted directly from definition; the construction of \( G^{-1}(2^k D_{n-k}^*) \) for the transformation \( G^{-1}(z) = 2 - z^{-1} \) is immediate. From the geometry it is clear that the condition \( D_{n+1}^* \subset G^{-1}(2^k D_{n-k}^*) \) is equivalent to the following inequality comparing the ratio of diameter to distance from the origin for the circles generating \( D_{n+1}^* \) and \( G^{-1}(2^k D_{n-k}^*) \) respectively:

\[
\frac{2}{2n + 1} \geq \frac{1}{2 - 2^{-k}} \left[ \left( 2 - \left( 2 \left( 1 + \frac{2}{2(n-k)-1} \right)^{-1} \right) - (2 - 2^{-k}) \right) \right].
\]
In the exceptional case $k = n$, the region $2^k D_{n-k}^*$ is simply the circle $|z| < 2^n$, and $G^{-1}(2^k D_{n-k}^*)$ is the domain $|z - 2| > 2^{-n}$. Reasoning as before we find that the condition $D_{n+1}^* \subseteq G^{-1}(2^k D_{n-k}^*)$ for $k = n$ is indeed equivalent to the $k = n$ case of the inequality (7). Thus the proof of the lemma has reduced to proving the inequality (7) for $n = 0, 1, 2, \cdots$ with $k = 0, 1, \cdots, n$. Simplifying (7), we obtain the following equivalent forms:
\[
\frac{2}{2n + 1} \geq \frac{1}{2^{k+1} - 1} \left[ 1 - \frac{2(n - k) - 1}{2(n - k) + 1} \right] = \frac{2}{(2^{k+1} - 1)(2(n - k) + 1)},
\]

\[
(2^{k+1} - 1)(2(n - k) + 1) \geq 2n + 1,
\]

\[
(2^{k+1} - 2) \cdot 2(n - k) + (2^{k+1} - 2k - 2) \geq 0.
\]

Since in the last form both terms on the left are clearly non-negative for admissible values of \( n \) and \( k \), the desired inequality is established. This completes the proof of the lemma.

Returning to the proof of the theorem proper, recall that for the domains \( Q_n \) of (6), \( Q_{n+1} \supset Q_n \) and \( \bigcup_{n=0}^{\infty} Q_n \) omits only the half-line \([1, \infty]\). Since the principal star domain \( M_f \) is starlike, \( z \in M_f \) and \( \xi \in \mathbb{C}M_f \) implies \( (z/\xi) \in [1, \infty] \). Thus for a closed and bounded set \( T \) contained in \( M_f \), the union of the origin and all points of the form \( z/\xi \) for \( z \in T \) and \( \xi \in \mathbb{C}M_f \) forms a closed and bounded set \( V \) not intersecting \([1, \infty]\). Clearly for \( n \) sufficiently large, \( V \subset Q_n \). But for \( z \in T: \xi \in \mathbb{C}M_f \) implies \( (z/\xi) \in V \subset Q_n \), or in other words, \( z \in Q_n \); and thus \( z \in P_{Q_n} \). Therefore \( T \) is a closed and bounded set in \( P_{Q_n} \). Recalling the lemma and the fact that \( W_n \{1\}_k = 1 \), and applying Theorem 6, the conclusion of Theorem 7 follows.

References


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