CARTESIAN PRODUCTS WITH INTERVALS

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1. Introduction.

Theorem. There exists a combinatorial n-manifold $M^n$ with boundary $B$ such that $M^n \times I = I^{n+1}$ and $\pi_1(B) \neq 1$ if and only if $n \geq 4$.

It has been proved by Bing [1] that such manifolds cannot exist for $n < 4$. For $n = 4$ an example has been given by Poénaru [6], and we give here a construction which gives such an $M^n$ for all $n \geq 5$. The proof uses the affirmative solution of the generalized Poincaré Conjecture (for $n \geq 7$ by Stallings [7] and for $n = 5, 6$ by Zeeman [10]), and the generalized Schoenflies theorem.

2. Two consequences of the Stallings-Zeeman results.

Proposition 1. If $M^n$ is a contractible combinatorial n-manifold, $n \geq 5$, and the boundary of $M^n$ is $S^{n-1}$, then $M^n$ is homeomorphic with $I^n$.

Proof. Attach two copies of $M^n$ along $S^{n-1}$, using the identity map. The result is a combinatorial $n$-manifold $T^n$, and using van Kampen's theorem and duality we get that $T^n$ has the homotopy type of $S^n$. Since $n \geq 5$, $T^n$ is homeomorphic with $S^n$, and we note that $S^{n-1}$ is nicely embedded. By the Schoenflies theorem ([4] or [2]), it follows that $M^n$ is an $n$-cube.

Proposition 2. If $M^n$ is a contractible combinatorial $n$-manifold and $n \geq 5$, then $M^n \times I$ is homeomorphic with $I^{n+1}$.

Proof. Let $B$ be the boundary of $M^n$. Then the boundary of $M^n \times I$ is $T^n = (M^n \times \{0, 1\}) \cup (B \times I)$. Again it is routine to check that $T^n$ has the homotopy type of $S^n$, so we conclude that $T^n$ is an $n$-sphere. By Proposition 1 we have that $M^n \times I$ is homeomorphic with $I^{n+1}$.

3. Construction. Let $P$ be the 2-polyhedron defined by Newman in [5]. This polyhedron has $\pi_1(P) \neq 1$ whereas $H_1(P, Z) = 0 = H_2(P, Z)$. For $n \geq 5$ we embed $P$ in $S^n$ as a subcomplex and use Newman's result that $\pi_1(S^n - P) = 1$. Let $T$ be a regular neighborhood of $P$ with $P$ lying in the interior of $T$ (see [8, p. 293]). Then the boundary $B$ of

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2 These two propositions have also been noted by J. J. Andrews, Imbedding homotopy cells, to appear.
$T$ is an $(n-1)$-manifold which is also the boundary of $S^n - \text{Int. } T$. We define $M^n = S^n - \text{Int. } T$.

Now the natural deformation retraction of $T$ onto $P$ induces a map $\phi: B \to P$ and it is known that $\phi: \pi_1(B) \to \pi_1(P)$ is surjective [3; 5]. Hence $\pi_1(B) \neq 1$, and to complete our construction we need only to show that $M^n$ is contractible and apply Proposition 2.

Now $\pi_1(M^n) = \pi_1(S^n - P)$ since $S^n - P$ may be deformation retracted onto $M^n$, and it follows that $\pi_1(M^n) = 1$. Since $T$ has the same homotopy type as $P$, it is homologically trivial. By duality, $M^n$ is homologically trivial and hence contractible.

References

1. R. H. Bing, A 3-cell is the only object whose cartesian product with an arc is a 4-cell, Abstract 564-257, Notices Amer. Math. Soc. vol. 7 (1960) p. 68.