

LOCALIZATION IN A GRADED RING

J. H. DE BOER¹

If one wants to investigate the properties and relations of homogeneous ideals in a commutative graded ring, one has as a model on the one hand the well known case of a polynomial ring and on the other hand the general commutative ideal theory. The case of a polynomial ring has been studied for the sake of algebraic geometry, and one of the methods was traditionally the passage to nonhomogeneous coordinates by choosing a suitable hyperplane of infinity [5, pp. 750–755; 7, pp. 491–496]. On the other hand it appears that the homogeneous ideals of a graded ring form a system that is closed under the usual ideal operations, as is the system of all ideals of a commutative ring. Thus one may try to copy the whole ideal theory, but now for homogeneous ideals (and homogeneous elements) only.

Samuel [4], Northcott [3] and Yoshida [6] have proved the elementary properties of homogeneous ideals for a graded ring. In this paper we investigate how far the process of localization can be carried over to graded rings. The degrees in our graded ring are the integers; the case of a bigraded ring is each time treated as a corollary.

In §1 we summarize the elementary properties of homogeneous ideals. Having formulated and proved Lemma 1, all proofs become straightforward. In §2 we study the localization (i.e., passage to a ring of fractions) with respect to a prime ideal or a finite set of prime ideals. Here we introduce the concept of a relevant prime ideal, as did Yoshida. In §3 we discuss the transition to a nonhomogeneous ring by choosing a hyperplane of infinity. This may be called localization in the sense of the Zariski topology. By a hyperplane of infinity we mean simply a homogeneous element l of degree one, which is not nilpotent. The corresponding nonhomogeneous ring can be obtained in two ways, namely as $R/(l-1)R$, but also as follows: Let $[l]$ be the multiplicatively closed subset of R , consisting of all powers of l . Then the ring of fractions $R_{[l]}$ is again a graded ring. The zero-degree subring $R_{[l]0}$ of $R_{[l]}$ is our nonhomogeneous ring, i.e., $R/(l-1)R \cong R_{[l]0}$. As elements of degree one may happen to be

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scarce (already if R is a polynomial ring over a finite field), a generalization is given where the pair $(l, 1)$ is replaced by a pair of homogeneous elements (l, m) with $\deg l - \deg m = 1$.

1. Basic properties. By a graded ring we mean a commutative ring R , containing 1, in which a family of additive subgroups R_i , i running through the group Z of integers, is specified such that

- (i) as additive group, R is the (weak) direct sum $\sum_{i=-\infty}^{\infty} R_i$,
- (ii) $R_i R_k \subset R_{i+k}$ for any $i, k \in Z$.

The elements of R_i are said to be homogeneous of degree i . (i) means that any $x \in R$ has a unique expression $x = \sum x_i$ as a finite sum of homogeneous elements. The x_i are called the homogeneous constituents of x .

Similarly, a bigraded ring R is a direct sum $\sum R_{ij}$ of additive subgroups R_{ij} (i and j running through Z), such that $R_{ij} R_{kl} \subset R_{i+k, j+l}$. The elements of R_{ij} are called homogeneous of degree (i, j) . An example of a graded, respectively bigraded ring is a polynomial ring $k[x_0, \dots, x_m]$, respectively $k[x_0, \dots, x_m; y_0, \dots, y_n]$ over a field k , where x_0, \dots, y_n are indeterminates.

Let R be a graded ring. The following two conditions on an ideal \mathfrak{a} of R are equivalent:

- (i) \mathfrak{a} is generated by a set of homogeneous elements,
- (ii) the homogeneous constituents of any element of \mathfrak{a} belong to \mathfrak{a} .

If \mathfrak{a} satisfies this condition, it is called a homogeneous ideal. In other words a homogeneous ideal is compatible with the grading. The residue class ring of a graded ring by a homogeneous ideal is again a graded ring.

Let S be a subset of R , then the subset S' of S that consists of all homogeneous elements of S , is called the homogeneous part of S . If \mathfrak{a} is an ideal of R , the ideal \mathfrak{a}_H generated by the homogeneous part of \mathfrak{a} is the maximal homogeneous ideal contained in \mathfrak{a} . It will cause no confusion if we call \mathfrak{a}_H also the homogeneous part of \mathfrak{a} .

We shall often need the phrase: let $x_{h+1} + \dots + x_{h+n}$ be the decomposition of x into its homogeneous constituents, where the subscript denotes the degree. We shall abbreviate this by saying: let $x = x_{h+1} + \dots + x_{h+n}$.

If x, y are elements of R , the homogeneous constituents of xy are somewhat complicated. The following lemma seems to be a natural way of bringing a part of the information $xy=0$ into a convenient homogeneous form.

LEMMA 1. *If $xy^f = 0$ for some positive exponent f , then there is a positive integer g such that $xy^g = 0$ for all homogeneous constituents y_i of y .*

PROOF. Let $x = x_{h+1} + x_{h+2} + \dots$ and $y = y_{k+1} + y_{k+2} + \dots$.

(1) We shall first show that $xy^g_{k+1} = 0$ for $g \gg 1$. Suppose we have proved that $x_{h+1}y^g_{k+1} = x_{h+2}y^g_{k+1} = \dots = x_{h+m}y^g_{k+1} = 0$. Then $xy^f = 0$ becomes, after multiplying by y^g_{k+1} ,

$$(x_{h+m+1} + \dots)y^g_{k+1}y^f = 0,$$

hence $x_{h+m+1}y^{g+f}_{k+1} = 0$.

(2) Suppose now that we have proved, for $j = k+1, \dots, k+n$ that $xy^g_j = 0$ if $g \gg 1$. If we use the fact that $A - B$ is a factor of $A^g - B^g$, we have

$$0 = x(y_{k+1} + \dots + y_{k+n})^g - (-)^g(y_{k+n+1} + \dots)^g,$$

hence $0 = x(y_{k+n+1} + \dots)^{g_1}$ for $g_1 \gg 1$, hence by step (1), $xy^{g_2}_{k+n+1} = 0$ for $g_2 \gg 1$.

This lemma remains true in the case of a bigraded ring, as we see by applying the previous case twice. Hence all other statements of this section hold for a bigraded ring as well.

In the next theorem we dispose to some extent of the inhomogeneous elements of R .

THEOREM 1 [3]. (i) *The element 1 is homogeneous of degree 0.*

(ii) *If $xy = 0 \Rightarrow x = 0$ or $y = 0$ for any two homogeneous elements x, y , then R is an integral domain.*

(iii) *If for any two homogeneous elements x, y*

$$xy = 0 \Rightarrow x = 0 \text{ or } y^g = 0 \text{ for some exponent } g,$$

then (0) is a primary ideal.

[We note that (ii) gives the criterion for a homogeneous ideal to be prime in terms of homogeneous elements. Similarly (iii) gives the criterion for any homogeneous ideal to be primary.]

PROOF. (i) Let x_0 be the constituent of degree 0 of 1. Then, by the grading, if y is any homogeneous element, $1 \cdot y = x_0y$, hence $1 - x_0$ acts (by multiplication) trivially on the homogeneous elements, hence on all elements, hence on 1, i.e., $1 - x_0 = 0$.

(ii) is easy. To prove (iii) suppose that $xy = 0$, where $x = x_{h+1} + \dots$ and $y = y_{k+1} + \dots$ are any elements of R . By Lemma 1, $x_i y^g_j = 0$ for all i, j and $g \gg 1$. Hence, by assumption, either all x_i are zero, or for $g \gg 1$, $y^g_j = 0$ for all j . Hence $x = 0$ or $y^g = 0$ for $g \gg 1$.

THEOREM 2 [6; 3]. *If a, b, q are ideals, we have*

(i) $(a \cap b)_H = a_H \cap b_H,$

(ii) $(\text{Rad } a)_H = \text{Rad } (a_H),$

(iii) \mathfrak{q} is primary $\Rightarrow \mathfrak{q}_H$ is primary.

If \mathfrak{a} , \mathfrak{b} are homogeneous, then so are (iv) $\mathfrak{a} + \mathfrak{b}$, (v) $\mathfrak{a}\mathfrak{b}$, (vi) $\mathfrak{a} \cap \mathfrak{b}$, (vii) $\mathfrak{a} : \mathfrak{b}$ and (viii) $\text{Rad } \mathfrak{a}$.

(ix) If \mathfrak{a} is homogeneous and intersection of a finite number of primary ideals, then \mathfrak{a} has an irredundant primary decomposition $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$, where all \mathfrak{q}_p are homogeneous [1, p. 198].

PROOF. (i) is trivial. To prove (viii), let $x = x_{h+1} + \cdots \in \text{Rad}(0)$, i.e., $x^f = 0$ for some f . Hence $x_i^f = 0$ for any i and $g \gg 1$, by Lemma 1. Thus all $x_i \in \text{Rad}(0)$.

Now we prove (ii). If h is a homogeneous element in $\text{Rad } \mathfrak{a}$, then $h^f \in \mathfrak{a}$ for some f , hence $h^f \in \mathfrak{a}_H$. Conversely, since $\mathfrak{a}_H \subset \mathfrak{a}$, we have $\text{Rad}(\mathfrak{a}_H) \subset \text{Rad } \mathfrak{a}$, hence $\text{Rad}(\mathfrak{a}_H) \subset (\text{Rad } \mathfrak{a})_H$ since $\text{Rad}(\mathfrak{a}_H)$ is homogeneous.

We prove (iii) with the criterion of Theorem 1(iii). Let x, y be homogeneous elements such that $xy \in \mathfrak{q}_H$ and $x \notin \mathfrak{q}_H$. Then $y^g \in \mathfrak{q}$ for $g \gg 1$, hence $y^g \in \mathfrak{q}_H$.

(iv), (v), (vi), (vii) are easy. As for (ix), if $\mathfrak{a} = \mathfrak{q}'_1 \cap \cdots \cap \mathfrak{q}'_n$ is an irredundant primary decomposition of \mathfrak{a} , then $\mathfrak{a}_H = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$, where $\mathfrak{q}_p = \mathfrak{q}'_p$ is primary, by (i) and (iii). If \mathfrak{a} is homogeneous, then $\mathfrak{a} = \mathfrak{a}_H$ and the result follows. [Hence, if \mathfrak{q}'_p is \mathfrak{p}_p -primary then \mathfrak{p}_p is homogeneous since the prime ideals of \mathfrak{a} are uniquely determined, and furthermore \mathfrak{q}_p is also \mathfrak{p}_p -primary by (ii).]

2. Localization.

THEOREM 3. Let S be a (nonempty) multiplicatively closed subset of R and \mathfrak{a} a homogeneous ideal. Then $\mathfrak{a}_S = \{x; xs \in \mathfrak{a} \text{ for some } s \in S\}$ is also a homogeneous ideal.

PROOF. If $xs \in \mathfrak{a}$, then by Lemma 1, $x_i s^g \in \mathfrak{a}$ for any homogeneous constituent x_i of x and $g \gg 1$. But $s^g \in S$.

A homogeneous prime ideal \mathfrak{p} is called relevant if for any $g \gg 1$ there is a homogeneous element of degree g , which is not contained in \mathfrak{p} . All other homogeneous prime ideals we shall optimistically call irrelevant. (In this way we introduce an asymmetry, i.e., we privilege the positive degrees above the negative ones. We might, however, as well favour the negative degrees.) In a homogeneous polynomial ring R the only irrelevant prime ideal is $R_1 + R_2 + \cdots$. That a homogeneous prime ideal \mathfrak{p} is irrelevant means that the positive degrees of R/\mathfrak{p} which really appear, have a common divisor $\neq 1$.

For a bigraded ring we have the same concept: \mathfrak{p} is a relevant homogeneous prime ideal if there is a homogeneous element of degree (f, g) outside \mathfrak{p} , for any sufficiently high f and g .

THEOREM 4 [6, p. 46, Lemma 2]. *If $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are relevant homogeneous prime ideals and if \mathfrak{a} is a homogeneous ideal which is not contained in any of the \mathfrak{p}_i , then for any sufficiently high degree g there is a homogeneous element of degree g that is contained in \mathfrak{a} , but in none of the \mathfrak{p}_i .*

PROOF. We may suppose that there is no inclusion relation between the \mathfrak{p}_i (by omitting any prime ideal that is not maximal in their set). Then for any $i=1, \dots, n$ we have $\mathfrak{a}\mathfrak{p}_1 \cdots \mathfrak{p}_{i-1}\mathfrak{p}_{i+1} \cdots \mathfrak{p}_n \subset \mathfrak{p}_i$. Hence there is a homogeneous element $a_i \in \mathfrak{a}\mathfrak{p}_1 \cdots \mathfrak{p}_{i-1}\mathfrak{p}_{i+1} \cdots \mathfrak{p}_n$ which is not in \mathfrak{p}_i . We may multiply a_i by a homogeneous element outside \mathfrak{p}_i to get an element with the same properties as a_i , but with any sufficiently high degree. In particular, if we make all a_i of the same degree, then $\sum a_i$ is a homogeneous element of \mathfrak{a} which does not belong to any of the given prime ideals.

REMARK 1. The assumption that the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are homogeneous is inessential, since if the \mathfrak{p}_i are not homogeneous we can replace them without damage by their homogeneous parts (these still being assumed to be relevant).

REMARK 2. A weaker version of Theorem 4 may be formulated. Instead of assuming the \mathfrak{p}_i to be relevant, we assume only that, for any i , the complement S_i of \mathfrak{p}_i in R contains a homogeneous element of degree >0 . Then the elements a_i of the proof may first be assumed to have positive degrees and then, by taking suitable powers of them, they may be assumed to have a common degree. Thus we reach the conclusion that a homogeneous element $a \in \mathfrak{a}$ exists of some positive degree, that is in none of the \mathfrak{p}_i .

THEOREM 5. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be homogeneous prime ideals. Assume that, for any i , the complement S_i of \mathfrak{p}_i in R has a homogeneous element of positive degree. Let S be the intersection of the S_i and S' the homogeneous part of S . Then for any homogeneous ideal \mathfrak{a} we have $\mathfrak{a}_{S'} = \mathfrak{a}_S$.*

PROOF. Obviously $\mathfrak{a}_{S'} \subset \mathfrak{a}_S$ and, by Theorem 3, \mathfrak{a}_S is also homogeneous. Let x be a homogeneous element of \mathfrak{a}_S . Then $\mathfrak{a}:x$ is homogeneous and not contained in any of the \mathfrak{p}_i . Hence, by Theorem 4, Remark 2, there is a homogeneous element s' outside the \mathfrak{p}_i , contained in $\mathfrak{a}:x$, i.e., $s' \in S'$ and $xs' \in \mathfrak{a}$.

It follows that, in the situation of the last theorem, if we are only interested in the homogeneous ideals, we may replace R_S by $R_{S'}$, which is again a graded ring.

Theorems 4 and 5 give no difficulty for a bigraded ring.

3. Choosing a hyperplane of infinity. Let $R = \sum R_i$ be a graded

ring. The homogeneous divisors of one (=homogeneous units) form a multiplicative subgroup U of R . It is the largest multiplicative subgroup consisting of homogeneous elements. Let G be the group of the degrees of the elements of U . We are especially interested in the case that G coincides with the whole group Z , i.e., the case that R contains a homogeneous unit u of degree 1. Then $uu^{-1}=1$, $\deg u=1$, $\deg u^{-1}=-1$ and the structure of R is determined by R_0 and the knowledge that there is such a homogeneous unit u of degree 1. In fact, all groups $R_i=u^iR_0$ are isomorphic and the multiplication in R is obtained from that in R_0 as follows: if $x_i \in R_i$, $x_j \in R_j$, then $x_ix_j=u^{i+j}(u^{-i}x_i)(u^{-j}x_j)$. Also any ideal \mathfrak{a}_0 of R_0 is the restriction of the homogeneous ideal $R\mathfrak{a}_0$ of R . Thus the homogeneous ideals of R and their operations correspond uniquely to the restrictions in R_0 . For instance, if \mathfrak{q}_0 is a primary ideal of R_0 , then $R\mathfrak{q}_0$ is a homogeneous primary ideal of R , by the criterion of Theorem 1 (iii). Hence, instead of the whole R , we need only consider R_0 .

Now consider the homomorphism $\lambda: R \rightarrow R_0$ which assigns to the element $x_i \in R_i$ the element $u^{-i}x_i \in R_0$. As $\lambda x = \lambda(ux)$ for any $x \in R$, the kernel of λ contains $(u-1)R$. Also, for any $x = \sum x_i \in R$, we have $x = \sum u^{-i}x_i + \sum (u^i-1)(u^{-i}x_i)$, hence the kernel of λ is $(u-1)R$. Thus we have the exact sequence of (nongraded) R -modules

$$0 \rightarrow (u-1)R \rightarrow R \xrightarrow{\lambda} R_0 \rightarrow 0,$$

which splits. The projection λ induces a 1-1 correspondence of the set of homogeneous ideals \mathfrak{a} of R onto the set of ideals of R_0 , or onto the set of ideals \mathfrak{a}' of R containing $u-1$. This correspondence is given by $\mathfrak{a}' = \mathfrak{a} + (u-1)R$, $\mathfrak{a} = (\mathfrak{a}')_H$.

For example, let S be the homogeneous part of the complement of a relevant homogeneous prime ideal (or of a finite number of such prime ideals) of a graded ring R , then (by Theorem 4 in the case of more than one prime ideal) there are elements of S of any sufficiently high degree. Hence R_S contains units of any degree and it is sufficient to consider the zero-degree part $(R_S)_0$ of R_S .

Let now l be a homogeneous element of degree 1 in R , such that $l \notin \text{Rad}(0)$. We take the multiplicative set $[l] = \{l^f; f=0, 1, \dots\}$. Then $R_{[l]}$ is a graded ring in which the image of l is a unit of degree 1. Hence we need only consider $R_{[l]_0}$. This is "the inhomogeneous ring, derived from R by choosing l as hyperplane of infinity."

The kernel $N = \{x; x \in R, l^g x = 0 \text{ for some } g\}$ of the natural homomorphism $\phi: R \rightarrow R_{[l]}$ is contained in $(l-1)R$. In fact, if $l^g x = 0$, then $x = (1-l^g)x \in (l-1)R$. Hence the kernel of the composed homomor-

phism $\lambda\phi: R \rightarrow R_{[l]_0}$ is $(l-1)R$. Moreover, $\lambda\phi$ is onto $R_{[l]_0}$; in the case that l is not a zero divisor, this is obvious; in the general case, it follows from the fact that, by definition, $R_{[l]} = \bar{R}_{[\bar{l}]}$, where $\bar{R} = R/N$ and $\bar{l} = l/N$. Thus the inhomogeneous ring corresponding to l may also be defined as the residue class ring $R/(l-1)R$. By the properties of passing to a ring of fractions, we have a 1-1 correspondence between the ideals \mathfrak{a}' of $R_{[l]_0}$ and the homogeneous ideals \mathfrak{a} of R for which $\mathfrak{a} : l = \mathfrak{a}$. We shall formulate the result as follows:

THEOREM 6 [2, p. 428, Lemma 1.1]. *The two sets of ideals in R :*

$$A = \{\mathfrak{a}; \mathfrak{a} \text{ homogeneous, } \mathfrak{a} : l = \mathfrak{a}\},$$

$$A' = \{\mathfrak{a}'; l - 1 \in \mathfrak{a}'\},$$

are in 1-1 correspondence by:

$$\mathfrak{a}' = \mathfrak{a} + (l-1)R,$$

$$\mathfrak{a} = (\mathfrak{a}')_H.$$

This correspondence is an isomorphism for the ideal operations (i) $\mathfrak{a} \cap \mathfrak{b}$ (hence also for the relation $\mathfrak{a} \subset \mathfrak{b}$), (ii) $\mathfrak{a} : \mathfrak{b}$, (iii) $\text{Rad } \mathfrak{a}$ and for the property: (iv) \mathfrak{q} is a primary ideal.

l may be a divisor of zero. In the algebraic geometry case, where $R = k[x_0, \dots, x_n]/\mathfrak{a}$ is the residue class ring of a polynomial ring by a homogeneous ideal \mathfrak{a} , this means that some component of the algebraic point set corresponding to \mathfrak{a} , lies in the hyperplane $l=0$.

It may happen, however, that for a nonirrelevant prime ideal \mathfrak{p} there does not exist a homogeneous element l of degree 1 which does not belong to \mathfrak{p} , and hence that we are not able to represent in this way \mathfrak{p} in a nonhomogeneous ring. It may even happen that $R_1 = (0)$. We shall therefore generalize the procedure.

First, we may assume that the degrees, that really occur, have no common factor (by dividing all degrees by their greatest common divisor). Suppose there are two homogeneous elements $l, m \in R$, $l \neq 0$, $m \neq 0$, such that $\deg l - \deg m = 1$. (This, for instance, is always the case if R is an integral domain.) Then, if $[l, m]$ stands for the set $\{l^g m^g; g = 0, 1, 2, \dots\}$ and if $0 \notin [l, m]$, we can reduce the graded ring $R_{[l, m]}$ to its zero-degree part $R_{[l, m]_0}$. Moreover, $R_{[l, m]}$ can be written as $(R_{[m]})_{[\bar{l}/\bar{m}]}$, where \bar{l} and \bar{m} are the natural images of l and m in $R_{[m]}$. For simplicity, we shall denote these natural images again by l and m and thus write $(R_{[m]})_{[l/m]}$ instead of $(R_{[m]})_{[\bar{l}/\bar{m}]}$. By the previous case, we have $(R_{[l, m]_0}) = R_{[m]} / ((l/m) - 1)R_{[m]} = R_{[m]} / (l - m)R_{[m]}$. Thus:

THEOREM 7. *Let l, m be homogeneous elements of the graded ring R , such that $\deg l - \deg m = 1$ and $0 \notin [l, m]$, then*

$$R_{[l,m]_0} = R_{[m]} / (l - m)R_{[m]}.$$

The ideals of this inhomogeneous ring correspond 1-1 with the homogeneous ideals of R that are prime to lm . Equivalently, the two sets of ideals of R ,

$$A = \{ \alpha; \alpha \text{ homogeneous, } \alpha : lm = \alpha \}$$

and

$$A' = \{ \alpha'; l - m \in \alpha', \alpha' : m = \alpha' \}$$

are in 1-1 correspondence, compatible with the ideal operations \cap, \cdot, Rad . This correspondence is given by

$$\begin{aligned} \alpha' &= \{ x; m^g x \in \alpha + (l - m)R \text{ for some } g \}, \\ \alpha &= (\alpha')_H. \end{aligned}$$

Note the symmetry between l and m in these considerations, for instance, $R_{[m]} / (l - m)R_{[m]} = R_{[l]} / (l - m)R_{[l]}$. We shall call (l, m) a *divisor of degree 1*.

Now we have achieved that any finite number of relevant homogeneous prime ideals can be simultaneously represented in some inhomogeneous ring (Theorem 4). In particular, let $R = k[x_0, \dots, x_n]$ be a polynomial ring over a finite field k . Then there are only a finite number of hyperplanes. If now α is a homogeneous ideal of R such that the irrelevant prime ideal $Rx_0 + \dots + Rx_n$ is not a prime ideal of α , then it may still happen that there is a component of α in each hyperplane, i.e., $\alpha : l \neq \alpha$ for each $l \in R_1$. In that case we cannot choose a hyperplane l of infinity such that α is properly represented in the corresponding inhomogeneous ring. But we can choose, instead, a "divisor (l, m) of degree 1", which "does not contain any component of α ".

If R is a bigraded ring, we can proceed similarly. We first assume that the homogeneous units of R have all possible degrees. Then the structure of R is determined by R_{00} (and the knowledge of the existence of units of any degree). Let u_1, u_2 be homogeneous units such that $\deg u_1 = (1, 0), \deg u_2 = (0, 1)$. By applying the above case twice, we have $R_{00} = R / (R(u_1 - 1) + R(u_2 - 1))$.

Let now l_1, m_1, l_2, m_2 be homogeneous elements such that $\deg l_1 - \deg m_1 = (1, 0), \deg l_2 - \deg m_2 = (0, 1)$. Let S be the set $[l_1, m_1, l_2, m_2]$ of all the products of these four elements and suppose that $0 \notin S$. Then

$$(R_S)_{00} = R_{[m_1, m_2]} / (R_{[m_1, m_2]}(l_1 - m_1) + R_{[m_1, m_2]}(l_2 - m_2))$$

is the inhomogeneous ring obtained by choosing the divisors (l_1, m_1) and (l_2, m_2) as first and second "hyperplane of infinity". The ideals of $(R_S)_{00}$ correspond 1-1 with the homogeneous ideals \mathfrak{a} of R for which $\mathfrak{a} : l_1 m_1 l_2 m_2 = \mathfrak{a}$.

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TECHNOLOGICAL UNIVERSITY, EINDHOVEN, NETHERLANDS