A REMARK ON THE RITT ORDER OF ENTIRE FUNCTIONS DEFINED BY DIRICHLET SERIES

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1. Let $f(s)$ $(s=\sigma+it)$ be an entire function defined by an everywhere absolutely convergent Dirichlet series

$$\sum_{n=1}^{\infty} a_n \exp(\lambda_n s) \qquad (0 < \lambda_n < \lambda_{n+1} \to \infty).$$

The Ritt order is defined [1, p. 77] as

$$\rho = \limsup_{\sigma=\infty} [\log \log M(\sigma)/\sigma]$$

with

$$M(\sigma) = \underset{-\infty<t<\infty}{\text{l.u.b.}} \, |f(\sigma + it)| .$$

2. We will show that the Ritt theorem [1, p. 78], which gives the value of ρ in terms of the sequences $\{|a_n|\}$ and $\{\lambda_n\}$ under the restriction $\liminf_{n=\infty} [\lambda_n/\log n]>0$, can be improved as follows:

Let ρ_1 be defined as

$$\rho_1 = \limsup_{n=\infty} [\lambda_n \log \lambda_n/\log |a_n|^{-1}].$$

We will prove the following:

THEOREM. *If* $\sum_{n=1}^{\infty} a_n \exp(\lambda_n s)$ *is absolutely convergent for all* s, *then* $\rho \geqq \rho_1$, *and if, in addition, it is assumed that*

$$\lim_{n=\infty} [\lambda_n \log \lambda_n/\log n] = \infty \tag{1}$$

then $\rho=\rho_1$.

PROOF. With no changes, the first part of the proof given by Mandelbrojt [2, p. 217] shows that $\rho \geqq \rho_1$ if $\rho < \infty$, and the same conclusion obviously holds if $\rho = \infty$.

To completely establish the theorem it is now enough to prove that if (1) holds, then $\rho \leqq \rho_1$. This result can be obtained by a modification of the second part of Mandelbrojt's proof [2, p. 218], as follows.

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We can assume $\rho_1 < \infty$, since otherwise the conclusion is trivial. Then for every $\delta > 0$ there exists $n_0 = n_0(\delta)$ such that $|a_n| \leqq \exp[-\lambda_n \log \lambda_n/(\rho_1+\delta)]$ for all $n \geqq n_0$. Therefore, for every σ

$$M(\sigma) \leqq \sum_{n=1}^{n_0-1} |a_n| \exp(\lambda_n\sigma) + \sum_{n=n_0}^{\infty} \exp[-\lambda_n \log \lambda_n/(\rho_1+\delta) + \lambda_n\sigma]$$

because the series involved in the formula is easily seen to be convergent, since from (1) it follows that for every $\epsilon > 0$, the series $\sum_{n=1}^{\infty} \exp(-\epsilon\lambda_n \log \lambda_n)$ converges to a finite sum $\phi(\epsilon)$.

Now, for any choice of $\alpha \geqq 0$ and $\beta > 0$ such that $\alpha+\beta = 1/(\rho+\delta)$ we have

$$\begin{aligned} M(\sigma) &\leqq \sum_{n=1}^{n_0-1} |a_n| \exp(\lambda_n\sigma) + \max_{n\geqq 1} [\exp(-\alpha\lambda_n \log \lambda_n + \lambda_n\sigma)] \\ &\quad \cdot \sum_{n=n_0}^{\infty} \exp(-\beta\lambda_n \log \lambda_n) \\ &\leqq \sum_{n=1}^{n_0-1} |a_n| \exp(\lambda_n\sigma) + \phi(\beta) \max_{x\geqq 0} \exp(-\alpha x \log x + x\sigma) \\ &= \sum_{n=1}^{n_0-1} |a_n| \exp(\lambda_n\sigma) + \phi(\beta) \exp\{\alpha \exp[(\sigma - \alpha)/\alpha]\} \end{aligned}$$

and consequently $\rho = \limsup_{\sigma=\infty} [\log \log M(\sigma)/\sigma] \leqq 1/\alpha$, and since $\sup \alpha = 1/(\rho_1+\delta)$ and δ is arbitrarily small, it follows that $\rho \leqq \rho_1$ and the theorem is proved.

Finally, it should be noticed that the assumption of the sequence $\{\lambda_n\}$ being monotonic, implies that (1) holds if and only if the series $\sum \exp(-\epsilon\lambda_n \log \lambda_n)$ converges for every $\epsilon > 0$, and therefore the previous theorem is the best possible result that can be obtained by the present method of proof.

References

1. J. F. Ritt, *On certain points of the theory of Dirichlet series*, Amer. J. Math. vol. 50 (1928) pp. 73–86.

2. S. Mandelbrojt, *Dirichlet series*, The Rice Institute Pamphlet, vol. 31, no. 4, October 1944, pp. 159–272.

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