THE MONOTONE UNION OF OPEN \( n \)-CELLS
IS AN OPEN \( n \)-CELL

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In a research announcement \([2]\) B. Mazur indicated that modulo the Generalized Schoenflies Theorem, the following theorem could be proved:

"If the open cone over a topological space \( X \) is locally Euclidean at the origin, then it is topologically equivalent with Euclidean space."

Ronald Rosen \([3]\) has described an ingenious proof of this theorem based on the now known \([1]\) Generalized Schoenflies Theorem. In the present paper we prove a stronger theorem without employing the Generalized Schoenflies Theorem.

Definitions and notation. If \( Q \) is an \( n \)-cell then \( \dot{Q}, Q \) denote the interior and boundary of \( Q \), respectively. An \( n \)-annulus is a homeomorph of \( S^{n-1} \times [0,1] \). If \( S \) is an \((n-1)\)-sphere in an \( n \)-cell, then \( I(S) \) denotes the interior (complementary domain) of \( S \). If \( S_1, S_2 \) are \((n-1)\)-spheres in an \( n \)-cell and \( S_1 \subset I(S_2) \), then \([S_1, S_2]\) (or equivalently \([S_2, S_1]\)) denotes the set \( Cl[I(S_2)] - I(S_1) \). An \((n-1)\)-sphere \( S \) embedded in a space \( X \) is collared if there is a homeomorphism \( h \) of \( S^{n-1} \times [0,1] \) into \( X \) such that \( h(S^{n-1} \times 1/2) = S \). Finally \( B \), will denote the \( n \)-ball of radius \( r \) in \( E^n \) and centered at the origin.

Lemma 1. Let \( S \) be a collared \((n-1)\)-sphere in the interior of an \( n \)-cell \( Q \) such that \( Cl[I(S)] \) is an \( n \)-cell.\(^1\) Let \( h \) be a homeomorphism of \( Q \) upon itself such that \( S \subset I(h(S)) \) and \( h|U = 1 \) where \( U \) is a nonempty open subset of \( I(S) \). Then \( h(S) \) is a collared \((n-1)\)-sphere in \( \dot{Q} \), \( Cl[I(h(S))] \) is an \( n \)-cell and \([S, h(S)]\) is an \( n \)-annulus.

Proof. Let \( f \) be a homeomorphism of \( S^{n-1} \times [0,1] \) into \( \dot{Q} \) such that \( f(S^{n-1} \times 0) = S, f(S^{n-1} \times [0,1]) \cap h(S) = 0 \), and \( f(S^{n-1} \times [0,1]) \cap I(S) = 0 \). Evidently \( I(S) \cup f(S^{n-1} \times [0,1]) \) is an \( n \)-cell. Hence there is a homeomorphism \( g \) of \( Q \) upon itself such that:

\[
\begin{align*}
(1) & \quad g(S) \subset U, \\
(2) & \quad gf(S^{n-1} \times 1/2) = S, \\
(3) & \quad g \mid h(S) = 1.
\end{align*}
\]

Then

\(^1\) The results of \([1]\) make this last part of the hypothesis unnecessary.

812

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THE MONOTONE UNION OF OPEN $n$-CELLS

$g^{-1}hgf(S^{n-1} \times [0, 1/2]) = g^{-1}hg[f(S^{n-1} \times 0), f(S^{n-1} \times 1/2)]$

$= g^{-1}h[g(S), S]$

$= g^{-1}[g(S), h(S)]$

$= [S, h(S)].$

Hence $[S, h(S)]$ is an $n$-annulus. Obviously $h(S)$ is collared and $	ext{Cl} [I(h(S))]$ is an $n$-cell.

**Lemma 2.** Let $S$ be a collared $(n-1)$-sphere in the interior of an $n$-cell $Q$ such that $\text{Cl} [I(S)]$ is an $n$-cell. Suppose $M$ is a compact subset of $Q$. Then there is a collared $(n-1)$-sphere $S'$ in $Q$ such that $I(S) \supset M \cup S$, $\text{Cl} [I(S')]$ is an $n$-cell, and $[S, S']$ is an $n$-annulus.

**Proof.** We may suppose without loss of generality that $Q$ is the unit ball $B_1$ in $E^n$ and that $I(S)$ contains the origin. Let $\epsilon > 0$ be small enough so that $B_\epsilon \subset I(S)$ and $M \subset J_\epsilon$. Let $h$ be a homeomorphism of $B_1$ upon itself such that $h|_{B_{1/2}} = 1$ and $h(B_\epsilon) \subset B_{1-\epsilon}$. Then $S' = h(S)$ contains $M \cup S$ in its interior. Lemma 1 insures that $h(S)$ is collared and that $[S, h(S)]$ is an $n$-annulus.

**Theorem.** Let $X$ be a topological space which is the union of a sequence $V_1 \subset V_2 \subset \cdots \subset V_i \subset \cdots$ of open subsets where each $V_i$ is homeomorphic to $E^n$. Then $X$ is homeomorphic to $E^n$.

**Proof.** Let $h_i$ map $E^n$ homeomorphically onto $V_i$. Then $h_i(B_1)$ is an $n$-cell in $V_i$. There is an integer $n_2$ such that

$$h_2(B_{n_2}) \supset h_1(B_2) \cup h_2(B_2).$$

Inductively, there is a sequence of integers $n_3, n_4, \ldots$, such that for all $i$,

$$h_i(B_{n_i}) \supset h_1(B_i) \cup \cdots \cup h_i(B_i) \cup h_{i-1}(B_{n_{i-1}}).$$

Since $X$ is locally Euclidean, $h_i(B_{n_i})$ is an $n$-cell in $X$ containing $h_{i-1}(B_{n_{i-1}})$ in its interior $h_i(B_{n_i})$. Finally $\bigcup_{i=1}^\infty B_{n_i} = X$. For if $x \in X$ there is an integer $j$ such that $x \in V_j$. Hence there is an integer $k > j$ such that $x \in h_j(B_k)$. But then $x \in h_k(B_n)$. Let $Q_i = h_i(B_n)$. Then $X = \bigcup_{i=1}^\infty Q_i$ where $Q_i$ is an $n$-cell, $Q_i \subset \tilde{Q}_{i+1}$, and $\tilde{Q}_{i+1}$ is open in $X$.

Let $S_1$ be a collared $(n-1)$-sphere in $\tilde{Q}_1$ such that $\text{Cl} [I(S_1)]$ is an $n$-cell. Applying Lemma 2 to the $n$-cell $Q_1$, we obtain a collared $(n-1)$-sphere $S_2$ in $\tilde{Q}_2$ such that $I(S_2) \supset Q_1 \cup S_1$, $[S_1, S_2]$ is an $n$-annulus, and $\text{Cl} [I(S_2)]$ is an $n$-cell. The same lemma applied to $Q_2$ and $S_2$ yields us a collared sphere $S_3$ in $\tilde{Q}_3$ such that $I(S_3) \supset Q_2 \cup Q_1$, $[S_2, S_3]$ is an $n$-annulus, and $\text{Cl} [I(S_3)]$ is an $n$-cell. Continuing this argument, we
get a sequence $S_1, S_2, \cdots$, of $(n-1)$-spheres such that $[S_i, S_{i+1}]$ is an $n$-annulus and $X = I(S_1) \cup [S_1, S_2] \cup [S_3, S_4] \cup \cdots$. Evidently $X$ is homeomorphic to $E^n$.

**References**


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**THREADS WITHOUT IDEMPOTENTS**

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If a thread $S$ has no idempotents and if $S^2 = S$, then $S$ is isomorphic with the real interval $(0, 1)$ under ordinary multiplication [2, Corollary 5.6]. Although the result is not nearly as pleasing as the special case just quoted, we shall give here a description of any thread without idempotents. Recall from [1] that a thread is a connected topological semigroup in which the topology is that induced by a total order.

First some examples. Let $X$ be a totally ordered set which is a connected space in the interval topology, let $T$ be a subset of $X$ containing, with $t$, all elements less than $t$, and let $\phi$ be any continuous function from $X$ into $(0, 1)$ whose restriction, $\phi_0$, to $T$ is a strictly order-preserving map of $T$ onto $(0, \alpha^2)$ where $\alpha = \text{l.u.b.} \phi(X)$. (We admit that $\alpha$ might be 1.) For such a $\phi$ to exist it is evidently necessary that $X$ not have a least element, that $T$ not have a greatest element and, provided $T \neq X$ so that the least upper bound, $q$, of $T$ exists, that $\phi(q) = \alpha^2$.

If $\phi(X)$ is the open interval $(0, \alpha)$, define a multiplication in $X$ by: $x \circ y = \phi^{-1}(\phi(x)\phi(y))$. With this definition it is quite easy to see that $X$ is a thread without idempotents and that $\phi$ is a homomorphism. In the event that $\phi(X)$ is the half closed interval $(0, \alpha]$ (which implies of course that $\alpha < 1$), put $A = \phi^{-1}(\alpha)$ and $B = \phi^{-1}(\alpha^2)$, observe that $q$ must be the least element of $B$, and let $\psi$ be any continuous

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