INTEGRALITY OF A CERTAIN KIND OF GENUS

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Introduction. The author introduced in [1] a new genus which is a generalization of index and A-genus. In this paper we shall give the complete proof of the integrality of our genus. However the integrality of A-genus has been proved in [2]. The author wishes to express his gratitude to Professor F. Hirzebruch for many valuable suggestions.

1. Let X^{4k} be a compact orientable differentiable $4k$-manifold. We defined the new genus by

$$\sum_{i=0}^{\infty} p_i = \prod_i (1+\gamma_i), \tag{1.1}$$

$$\prod_i \frac{(\gamma_i)^{1/2}}{\operatorname{tgh}(\gamma_i)^{1/2}}(1 + y \operatorname{tgh}^2(\gamma_i)^{1/2}) = \sum_{i=0}^{\infty} \Gamma_i(y, p_1, \cdots, p_i) \tag{1.2}$$

where p_i denotes the Pontryagin class of the dimension $4i$. The actual values of Γ_i's are as follows:

$$\begin{aligned} &\Gamma_0 = 1, \qquad \Gamma_1(y, p_1) = \left(y + \frac{1}{3}\right)p_1, \\ &\Gamma_2(y, p_1, p_2) = p_2y^2 + \frac{1}{3}(4p_2 - p_1^2)y + \frac{1}{45}(7p_2 - p_1^2), \\ &\Gamma_3(y, p_1, p_2, p_3) = p_3y^3 + \frac{1}{3}(6p_3 - p_1p_2)y^2 \\ &\qquad + \frac{1}{15}(17p_3 - 8p_1p_2 + 2p_1^3)y \\ &\qquad + \frac{1}{3^3\cdot 5\cdot 7}(62p_3 - 13p_1p_2 + 2p_1^3). \end{aligned} \tag{1.1}$$

We put

$$\Gamma_i(y, p_1, \cdots, p_i) = \sum_{j=0}^{i} \Gamma_{ij}(p_1, \cdots, p_i)y^j. \tag{1.4}$$

Our purpose is to prove that

$$\Gamma_{kj}(p_1, \cdots, p_k)[X^{4k}] = \text{integer} \tag{1.5}$$

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for every k and j. It suffices to show that

(i) $\Gamma_{kj}(p_1, \cdots, p_k)[X^{4k}]$ does not contain the factor 2 in its denominator when it is written as a quotient of relative prime integers.

(ii) $2^\alpha\Gamma_{kj}(p_1, \cdots, p_k)[X^{4k}]$ becomes an integer for a suitable integer α.

First of all let us prove (i). It is well known that

$$\frac{z^{1/2}}{\operatorname{tgh} z^{1/2}} = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}}{(2k)!} B_k z^k \tag{1.6}$$

[3, p.13]

and

$$\operatorname{tgh} z = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n z^{2n-1}. \tag{1.7}$$

Moreover it is known that

(1.8)
a. B_k (Bernoulli number) contains the factor 2 exactly to the first power in its denominator.
b. $(2k)!$ is not divisible by 2^{2k} [2, p. 341].

The statement (i) easily follows from these facts and (1.2).

2. Next let us prove (ii). It suffices to show this for the complex algebraic manifold [4]. In this case we have

$$\sum_{i=0}^{\infty} p_i = \prod_i (1 + \delta_i^2), \qquad \sum_{i=0}^{\infty} c_i = \prod_i (1 + \delta_i) \tag{2.1}$$

and

$$\begin{aligned} &\sum_{i=0}^{\infty} \Gamma_i(y, p_1, \cdots, p_i) \\ &= \prod_i \frac{\delta_i}{\operatorname{tgh} \delta_i} (1 + y \operatorname{tgh}^2 \delta_i) \\ &= \prod_i \frac{\delta_i}{1 - e^{-\delta_i}} \left(\frac{1}{\operatorname{tgh} \delta_i} + y \operatorname{tgh} \delta_i \right) (1 - e^{-\delta_i}) \\ &= \prod_i \frac{\delta_i}{1 - e^{-\delta_i}} \left\{ \frac{\frac{1}{2}(1 + e^{-2\delta_i})}{1 - \frac{1 - e^{-\delta_i}}{2}} + y \frac{\frac{1}{2}(1 - e^{-2\delta_i})(1 - e^{-\delta_i})}{1 - \frac{1 - e^{-2\delta_i}}{2}} \right\}. \end{aligned} \tag{2.2}$$

Hence we have

$$
\begin{aligned}
&\Gamma_k(y, p_1, \cdots, p_k) \\
&= \kappa_{4k}\Bigg[\prod_i \frac{\delta_i}{1-e^{-\delta_i}} \prod_i \Bigg[\frac{1}{2}(1+e^{-2\delta_i}) \\
&\quad \cdot \left\{1 + \frac{1-e^{-\delta_i}}{2} + \left(\frac{1-e^{-\delta_i}}{2}\right)^2 + \cdots + \left(\frac{1-e^{-\delta_i}}{2}\right)^{2k}\right\} \\
&\quad + \frac{1}{2} y(1-e^{-2\delta_i})(1-e^{-\delta_i}) \left\{1 + \frac{1-e^{-2\delta_i}}{2} + \left(\frac{1-e^{-2\delta_i}}{2}\right)^2 \right. \\
&\quad \left. + \cdots + \left(\frac{1-e^{-2\delta_i}}{2}\right)^{2k-2}\right\}\Bigg]\Bigg] \qquad (2.3) \\
&= \kappa_{4k}\Bigg[\Bigg(\sum_t y^t \sum_{a_1,\cdots,a_m} A^t_{a_1\cdots a_m} \sum_{i_1\neq\cdots\neq i_m} \exp(a_1\delta_{i_1} + \cdots + a_m\delta_{i_m})\Bigg) \\
&\quad \times \prod_i \frac{\delta_i}{1-e^{-\delta_i}}\Bigg]
\end{aligned}
$$

where A denotes some rational number which becomes an integer when it is multiplied by some power of 2 and a_i denotes some integer. Putting

$$
\prod_{i_1\neq\cdots\neq i_m} (1 + a_1\delta_{i_1} + \cdots + a_m\delta_{i_m}) = \sum_i d_i \qquad (2.4)
$$

we see that d_i is an integral cohomology class, i.e., $d_i \in H^{2i}(X^{4k}, Z)$. Hence we have

$$
\Gamma_{k_j}(p_1, \cdots, p_k)[X^{4k}] = \sum_i b_i T(X^{4k}, W_i) = \sum_i b_i \chi(X^{4k}, W_i) \qquad (2.5)
$$

where b_i denotes some rational number which becomes an integer by multiplying some power of 2 and $T(X^{4k}, W_i)$ denotes the Todd genus with regard to W_i and W_i denotes a complex-analytic vector bundle whose Chern class is $\sum d_i$ and $\chi(X^{4k}, W_i)$ denotes the Riemann-Roch number with regard to W_i [3, p.154]. Thus we have proved (ii).

References

1. Y. Tomonaga, *A generalization of Hirzebruch polynomial and cobordism decomposition*, Proc. Amer. Math. Soc. vol. 12 (1961) pp. 573–576.

2. A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces*. II, Amer. J. Math. vol. 31 (1959) pp. 315–382.

3. F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie*, Berlin, Springer-Verlag, 1956.

4. J. Milnor, *On the cobordism ring Ω^* and a complex analogue*, Amer. J. Math., to appear.

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