FUNCTIONS OF BVC TYPE\textsuperscript{1,2}

RICHARD E. HUGHS

Let \( f \) be a measurable function defined on the closed unit square \( Q = I \times I, I = [0, 1] \). For every \( x \in I \), let \( f_x \) be the function of \( y \) defined by \( f_x(y) = f(x, y) \) and for every \( y \in I \), let \( f^* \) be the function of \( x \) defined by \( f^*(x) = f(x, y) \). Let \( V(f_x) \) and \( V(f^*) \) be the variations of \( f_x \) and \( f^* \) on \( I \), respectively. The function \( f \) is said to be BVC (of bounded variation in the sense of Tonelli and Cesari \([1; 2]\)), if there are functions \( g \) and \( h \), which are equal to \( f \) almost everywhere on \( Q \), such that: 
\[
V(g_x) < \infty \text{ for almost all } x \in I, \ V(h^*_y) < \infty \text{ for almost all } y \in I, \\
\int_0^1 V(g_x) \, dx < \infty \text{ and } \int_0^1 V(h^*_y) \, dy < \infty.
\]
The purpose of this note is to show that if \( f \) is BVC, then there is a single function \( k \), which is equal to \( f \) almost everywhere on \( Q \), such that: 
\[
\int_0^1 V(k_x) \, dx < \infty \text{ and } \int_0^1 V(k^*_y) \, dy < \infty.
\]
This fact has already been established, \([3]\), in the special case where \( f \) is essentially linearly continuous.

Let \( f \) be a function defined on \([a, b]\),

\[ P : [a = \beta_0 < \beta_1 < \cdots < \beta_{m-1} < \beta_m = b] \]

be a partition of \([a, b]\), and define for \( x \in (\beta_{m-1}, \beta_m], (x \in [\beta_0, \beta_1] \text{ if } m = 1), m = 1, 2, \cdots, r\), the functions:

\[
\phi_P^+(f; x) = f(a) + \frac{1}{2} \sum_{i=1}^m \{ [f(\beta_i) - f(\beta_{i-1})] + |f(\beta_i) - f(\beta_{i-1})| \},
\]

\[
\phi_P^-(f; x) = \frac{1}{2} \sum_{i=1}^m \{ [f(\beta_i) - f(\beta_{i-1})] - |f(\beta_i) - f(\beta_{i-1})| \}
\]

and if \( 0 \leq j < k \leq r \), \( v(f; P; \beta_j, \beta_k) = \sum_{j=1}^k |f(\beta_i) - f(\beta_{i-1})| \). The functions \( \phi_P^+, \phi_P^- \) are monotone, nondecreasing. The norm of \( P \) is defined as \( |P| = \max [|\beta_i - \beta_{i-1}|, i = 1, 2, \cdots, r] \).

Lemma. If \( f \) is a BVC function on \([a, b]\) and \( \{P_n\} \) is a sequence of partitions of \([a, b]\), each a refinement of its predecessor with \( \lim_{n \to \infty} |P_n| = 0 \), then \( \lim_{n \to \infty} \phi_P^+(f; x) \) and \( \lim_{n \to \infty} \phi_P^-(f; x) \) exist at all points of \([a, b]\). If these limits are designated by \( \phi^+ \) and \( \phi^- \) respectively, then \( f = \phi^+ - \phi^- \) at all points of continuity of \( f \) and \( V(\phi^+ - \phi^-) \leq V(f) \).

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Proof. Let \( \{P_n\} \) be a sequence of partitions of \([a, b]\), where each is a refinement of its predecessor and \( \lim_{n \to \infty} |P_n| = 0 \). Let \( x \) be any number in \([a, b]\) and let \( (\alpha_n, \beta_n) \) be that subinterval of \( P_n \) which contains \( x \), (\([\alpha_n, \beta_n]\) if \( x = 0 \)).

Now,

\[
\phi^+_n(f; x) = f(a) + \frac{\left[ f(\beta_n) - f(a) + v(f; P_n; a, \beta_n) \right]}{2} = \frac{f(a)}{2} + \frac{f(\beta_n)}{2} + \frac{v(f; P_n; a, \alpha_n)}{2} + \frac{|f(\alpha_n) - f(\beta_n)|}{2}
\]

and

\[
\phi^-_n(f; x) = f(a)/2 - f(\beta_n)/2 + v(f; P_n; a, \alpha_n)/2 + |f(\beta_n) - f(\alpha_n)|/2.
\]

But, \( \{\alpha_n\} \) is a monotone, nondecreasing sequence, \( \{\beta_n\} \) is a monotone, nonincreasing sequence and \( \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = x \). Since \( f \) is BV, it follows that \( \lim_{n \to \infty} f(\beta_n) \) and \( \lim_{n \to \infty} |f(\beta_n) - f(\alpha_n)| \) exist, and \( \lim_{n \to \infty} v(f; P_n; a, \alpha_n) \) exists because \( \{v(f; P_n; a, \alpha_n)\} \) is a monotone, nondecreasing sequence which is bounded above by the variation of \( f \) on \([a, x]\). Thus, \( \lim_{n \to \infty} \phi^+_n(f; x) \) and \( \lim_{n \to \infty} \phi^-_n(f; x) \) exist for all \( x \in [a, b] \). Clearly, \( \lim_{n \to \infty} \left[ \phi^+_n(f; x) - \phi^-_n(f; x) \right] = \phi^+(x) - \phi^-(x) \) which equals \( f(x+) \) if \( x \neq \beta_n \) for any \( n = 1, 2, 3, \ldots \), and \( f(x) \) if \( x = \beta_n \) for some \( n = 1, 2, 3, \ldots \), which is just \( f(x) \) at all \( x \) which are points of continuity of \( f \). In either case, it is clear that \( V(\phi^+ - \phi^-) \leq V(f) \).

Theorem. Let \( f \) be a measurable function on the square, \( Q \), which is BV. Then, there is a single function, \( k \), equal to \( f \) almost everywhere on \( Q \), for which the sections \( k_x \) and \( k_y \) are BV almost everywhere in \( x \) and \( y \) respectively and \( \int_0^\infty V(k_x)dx < \infty \) and \( \int_0^\infty V(k_y)dy < \infty \).

Proof. Since \( f \) is BV on \( Q \), there are functions \( g \) and \( h \), equal to \( f \) almost everywhere, such that their sections \( g_x \) and \( h_y \) are BV almost everywhere in \( x \) and \( y \) respectively and for which \( \int_0^\infty V(g_x)dx < \infty \) and \( \int_0^\infty V(h_y)dy < \infty \).

Let \( \{P_n\} \) be a sequence of partitions of \( I \), each one a refinement of the previous one, with the following properties: \( \lim_{n \to \infty} |P_n| = 0 \); if \( P_n: [0 = \beta_0 < \beta_1 < \cdots < \beta_{n+1} = 1] \), then \( \beta_1, \ldots, \beta_n \) are such that \( G(x) = g(x, \beta^a_i) \) is summable for \( n = 1, 2, 3, \ldots \) and \( i = 1, 2, \cdots, r_n \); \( \lim_{n \to \infty} \beta_1 = 0 \); \( \lim_{n \to \infty} \beta_n = 1 \); and \( \beta \) is any element of \( P_n \) for all \( n = 1, 2, 3, \cdots \) for which \( g(x, \beta) \) is summable in \( x \).

For each \( n = 1, 2, 3, \cdots \), define, on the interval \([\beta_i, \beta_{i+1}] \subset I \), functions \( \phi^+_n(g_x, y) \) and \( \phi^-_n(g_x, y) \) exactly as described prior to the lemma where \( g_x \) is BV. Then, one defines:
Similarly, define \( \phi_{P_n}(x, y) \) if \( g_x \) is BV and let it be 0 if \( g_x \) is not BV.

Consider now, \( \phi_{P_n}^+(x, y) \) and \( \phi_{P_n}^-(x, y) \). If \( \beta_{j-1} < y \leq \beta_j \), \( 2 \leq j \leq r_n \), one has that

\[
\phi_{P_n}^+(x, y) = g_x(\beta_j)
\]

if \( g_x \) is BV, i.e. for almost all \( x \). But, since \( g(x, \beta_j) = g_x(\beta_j) \), \( i=1, 2, \ldots, r_n \) is a summable function of \( x \) for \( n = 1, 2, 3, \ldots \), it follows that \( \phi_{P_n}^+(x, y) \) is a measurable, and in fact summable, function of \( (x, y) \in \Omega \). Similarly \( \phi_{P_n}^-(x, y) \) is a measurable and summable function on \( \Omega \). Although the form of \( \phi_{P_n}^+ \) and \( \phi_{P_n}^- \) in the previous lemma, the only essential distinction is that instead of a \( g_x(\beta_i) \) term, there is a \( g_x(\beta_i) \) term appearing, where \( g_x \) is BV. Hence, by the lemma, limit \( n \to \infty \) \( \phi_{P_n}(x, y) \) and limit \( n \to \infty \) \( \phi_{P_n}^-(x, y) \) exist for all \( (x, y) \in \Omega \). Let \( g^+ \) and \( g^- \) be these limits, respectively, then \( g^+ \) and \( g^- \) are measurable since each is a limit of a sequence of measurable functions.

Suppose \( 0 < \alpha < 1 \) and \( 0 \leq x \leq 1 \). Then, there is \( N > 0 \) so that \( n > N \) implies \( P_n \) is such that \( \beta_j, \beta_{j+1} \leq \beta_i \leq \beta_n \). Hence, for \( n > N \), \( g_{P_n}^+(x, \alpha) - g_{P_n}^-(x, \alpha) \) is equal to \( \phi_{P_n}^+(x, \alpha) - \phi_{P_n}^-(x, \alpha) \) if \( g_x \) is BV, and \( g(x, \alpha) \) if \( g_x \) is not BV, and thus, by the lemma, one has that the limit, \( g^+(x, \alpha) - g^-(x, \alpha) \), is either \( g(x, \alpha) \) or \( g(x, \alpha^+) \) depending upon whether \( \alpha = \beta_j \) for some \( j \), \( n = 1, 2, 3, \ldots \) and \( g_x \) is BV. Thus, \( g^+(x, y) - g^-(x, y) = g(x, y) \) at all points \( (x, y) \in \Omega \) such that either \( g_x \) is not BV or \( g_x \) is BV and continuous at \( y \). Since a BV function can be discontinuous at no more than a countable number of points, if \( S \) is the set for which \( g^+ - g^- \) differs from \( g \), \( S \) is measurable since \( g^+ \), \( g^- \) and \( g \) are measurable, \( m(S_x) = 0 \) for all \( x \), where \( S_x = \{ y : (x, y) \in S \} \) and \( m_2(S) = \int_0^1 m(S_x) dx \), thus \( m_2(S) = 0 \). Hence, \( g^+ - g^- \) equals \( g \) almost everywhere on \( Q \) and \( g_{P_n}^+ \) and \( g_{P_n}^- \) are monotone for almost all \( x \in I \). Also, \( V(g_{P_n}^+ - g_{P_n}^-) \leq V(g_x) \).
It is clear from the definition of \( g^+ \) and \( g^- \), where \( \beta \in P_n \), \( n = 1, 2, 3, \ldots \), that since \( |g(x, \beta)| \leq |g(x, y)| + V(g_x) \), it follows that both \( |g^+(x, y)| \) and \( |g^-(x, y)| \) are bounded by \( |g(x, \beta)| + 2V(g_x) \) and since \( V(g_x) \) and \( g(x, \beta) \) are both summable on \( Q \), \( g^+ \) and \( g^- \) are also summable on \( Q \).

Let \((g^+)^s\) and \((g^-)^s\) be the integral means of \( g^+ \) and \( g^- \), i.e.,
\[
(g^+)^s(x, y) = \frac{1}{s^2} \int_{x-s}^{x+s} \int_{y-s}^{y+s} g^+(u, v) \, du \, dv,  \quad 0 \leq x, y \leq 1,
\]
and similarly for \( g^- \). It is clear that \((g^+)^s\) and \((g^-)^s\) are monotone for all \( x \) since \( g^+_x \) and \( g^-_x \) are monotone for almost all \( x \) and it is well known that \((g^+)^s\) and \((g^-)^s\) are continuous and converge almost everywhere on \( Q \) to \( g^+ \) and \( g^- \) as \( s \) goes to zero. Thus, \( k^+ = \lim \sup_{s \to 0} (g^+)^s \) and \( k^- = \lim \sup_{s \to 0} (g^-)^s \) have the same properties as \( g \) relative to \( f \); i.e., \( k = k^+ - k^- \) is equal almost everywhere to \( f \), \( k_x \) is BV for almost all \( x \), \( k^+_x \) and \( k^-_x \) are monotone and \( V(k_x) \leq V(g_x) \) for almost all \( x \). Thus,
\[
\int_0^1 V(k_x) \, dx < \infty.
\]

By exactly the same argument with \( h \), the same function \( k \) is obtained due to the symmetry of the integral means with respect to \( x \) and \( y \). Thus, there is a single function, \( k \), equal almost everywhere to \( f \), for which \( k_x \) and \( k_y \) are BV for almost all \( x \) and \( y \) respectively, and
\[
\int_0^1 V(k_x) \, dx < \infty \quad \text{and} \quad \int_0^1 V(k_y) \, dy < \infty.
\]

References


Purdue University