Let $f$ be a measurable function defined on the closed unit square $Q = I \times I$, $I = [0, 1]$. For every $x \in I$, let $f_x$ be the function of $y$ defined by $f_x(y) = f(x, y)$ and for every $y \in I$, let $f^y$ be the function of $x$ defined by $f^y(x) = f(x, y)$. Let $V(f_x)$ and $V(f^y)$ be the variations of $f_x$ and $f^y$ on $I$, respectively. The function $f$ is said to be BVC (of bounded variation in the sense of Tonelli and Cesari \cite{1,2}), if there are functions $g$ and $h$, which are equal to $f$ almost everywhere on $Q$, such that: $V(g_x) < \infty$ for almost all $x \in I$, $V(h^y) < \infty$ for almost all $y \in I$, $\int_Q V(g_x) \, dx < \infty$ and $\int_Q V(h^y) \, dy < \infty$. The purpose of this note is to show that if $f$ is BVC, then there is a single function $k$, which is equal to $f$ almost everywhere on $Q$, such that: $\int_Q V(k_x) \, dx < \infty$ and $\int_Q V(k^y) \, dy < \infty$. This fact has already been established, \cite{3}, in the special case where $f$ is essentially linearly continuous.

Let $f$ be a function defined on $[a, b]$, $P: [a = \beta_0 < \beta_1 < \cdots < \beta_{m-1} < \beta_r = b]$ be a partition of $[a, b]$, and define for $x \in (\beta_{m-1}, \beta_m]$ (or $x \in [\beta_0, \beta_1]$ if $m = 1$), $m = 1, 2, \ldots, r$, the functions:

$$
\phi^+_P(f; x) = f(a) + \frac{1}{2} \sum_{i=1}^{m} \left\{ \left[ f(\beta_i) - f(\beta_{i-1}) \right] + \left| f(\beta_i) - f(\beta_{i-1}) \right| \right\},
$$

$$
\phi^-_P(f; x) = -\frac{1}{2} \sum_{i=1}^{m} \left\{ \left[ f(\beta_i) - f(\beta_{i-1}) \right] - \left| f(\beta_i) - f(\beta_{i-1}) \right| \right\}
$$

and if $0 \leq j < k \leq r$, $v(f; P; \beta_j, \beta_k) = \sum_{i=j+1}^{k} \left| f(\beta_i) - f(\beta_{i-1}) \right|$. The functions $\phi^+_P, \phi^-_P$ are monotone, nondecreasing. The norm of $P$ is defined as $|P| = \max \left\{ |\beta_i - \beta_{i-1}|, i = 1, 2, \cdots, r \right\}$.

**Lemma.** If $f$ is a BV function on $[a, b]$ and $\{P_n\}$ is a sequence of partitions of $[a, b]$, each a refinement of its predecessor with $\lim_{n \to \infty} |P_n| = 0$, then $\lim_{n \to \infty} \phi^+_P(f; x)$ and $\lim_{n \to \infty} \phi^-_P(f; x)$ exist at all points of $[a, b]$. If these limits are designated by $\phi^+$ and $\phi^-$ respectively, then $f = \phi^+ - \phi^-$ at all points of continuity of $f$ and $V(\phi^+ - \phi^-) \leq V(f)$.
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Proof. Let \( \{ P_n \} \) be a sequence of partitions of \([a, b]\), where each is a refinement of its predecessor and \( \lim_{n \to \infty} |P_n| = 0 \). Let \( x \) be any number in \([a, b]\) and let \((\alpha_n, \beta_n)\) be that subinterval of \( P_n \) which contains \( x \), \(([\alpha_n, \beta_n] \) if \( x = 0\).

Now,

\[
\phi_{P_n}^+(f; x) = f(a) + \left[ f(\beta_n) - f(a) + v(f; P_n; a, \alpha_n) \right]/2
\]

\[
= f(a)/2 + f(\beta_n)/2 + v(f; P_n; a, \alpha_n)/2 + |f(\alpha_n) - f(\beta_n)|/2
\]

and

\[
\phi_{P_n}^-(f; x) = f(a)/2 - f(\beta_n)/2 + v(f; P_n; a, \alpha_n)/2 + |f(\beta_n) - f(\alpha_n)|/2.
\]

But, \( \{ \alpha_n \} \) is a monotone, nondecreasing sequence, \( \{ \beta_n \} \) is a monotone, nonincreasing sequence and \( \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = x \). Since \( f \) is BV, it follows that \( \lim_{n \to \infty} f(\beta_n) \) and \( \lim_{n \to \infty} |f(\beta_n) - f(\alpha_n)| \) exist, and \( \lim_{n \to \infty} v(f; P_n; a, \alpha_n) \) exists because \( \{ v(f; P_n; a, \alpha_n) \} \) is a monotone, nondecreasing sequence which is bounded above by the variation of \( f \) on \([a, x]\). Thus, \( \lim_{n \to \infty} \phi_{P_n}^+(f; x) \) and \( \lim_{n \to \infty} \phi_{P_n}^-(f; x) \) exist for all \( x \in [a, b] \). Clearly, \( \lim_{n \to \infty} [\phi_{P_n}^+(f; x) - \phi_{P_n}^-(f; x)] = \phi^+(f; x) - \phi^-(f; x) \) which equals \( f(x+) \) if \( x \neq \beta_n \) for any \( n = 1, 2, 3, \ldots \), and \( f(x) \) if \( x = \beta_n \) for some \( n = 1, 2, 3, \ldots \), which is just \( f(x) \) at all \( x \) which are points of continuity of \( f \). In either case, it is clear that \( V(\phi^+ - \phi^-) \leq V(f) \).

Theorem. Let \( f \) be a measurable function on the square, \( Q \), which is BVC. Then, there is a single function, \( k \), equal to \( f \) almost everywhere on \( Q \), for which the sections \( k_x \) and \( k_y \) are BV almost everywhere in \( x \) and \( y \) respectively and \( \int_0^1 V(k_x) \, dx < \infty \) and \( \int_0^1 V(k_y) \, dy < \infty \).

Proof. Since \( f \) is BVC on \( Q \), there are functions \( g \) and \( h \), equal to \( f \) almost everywhere, such that their sections \( g_x \) and \( h_y \) are BV almost everywhere in \( x \) and \( y \) respectively and for which \( \int_0^1 V(g_x) \, dx < \infty \) and \( \int_0^1 V(h_y) \, dy < \infty \).

Let \( \{ P_n \} \) be a sequence of partitions of \( I \), each one a refinement of the previous one, with the following properties: \( \lim_{n \to \infty} |P_n| = 0 \); if \( P_n: [0 = \beta_0^{a_n} \leq \beta_1^{a_n} < \beta_2^{a_n} < \cdots < \beta_{r_n}^{a_n} \leq \beta_{r_n+1}^{a_n} = 1] \), then \( \beta_1^{a_n}, \cdots, \beta_{r_n}^{a_n} \) are such that \( G(x) = g(x, \beta_i^{a_n}) \) is summable for \( n = 1, 2, 3, \ldots \) and \( i = 1, 2, \cdots, r_n \); \( \lim_{n \to \infty} \beta_i^{a_n} = 0 \); \( \lim_{n \to \infty} \beta_{r_n}^{a_n} = 1 \); and \( \beta \) is any element of \( P_n \) for all \( n = 1, 2, 3, \cdots \) for which \( g(x, \beta) \) is summable in \( x \).

For each \( n = 1, 2, 3, \cdots \), define, on the interval \( {[\beta_i^{a_n}, \beta_{i+1}^{a_n}] \subset I} \), functions \( \phi_{P_n}^+(g_x, y) \) and \( \phi_{P_n}^-(g_x, y) \) exactly as described prior to the lemma where \( g_x \) is BV. Then, one defines:
\[ g_p^+(x, y) = \begin{cases} 
\phi_p^+(g_x, \beta_i) & \text{if } y \in [\beta_1, \beta_n] \text{ and } g_x \text{ is BV}, \\
\phi_p^+(g_x, \beta_1) & \text{if } 0 \leq y \leq \beta_1 \text{ and } g_x \text{ is BV}, \\
\phi_p^+(g_x, \beta^n_r) & \text{if } \beta^n_r \leq y \leq 1 \text{ and } g_x \text{ is BV}, \\
g(x, y) & \text{if } g_x \text{ is not BV}. 
\end{cases} \]

Similarly, define \( g_{\overline{p}}^+(x, y) \) if \( g_x \) is BV and let it be 0 if \( g_x \) is not BV.

Consider now, \( g_p^+ \) and \( g_{\overline{p}}^+ \). If \( \beta_{j-1} \leq y < \beta_j \), \( 2 \leq j \leq r_n \), one has that
\[
gp(x, y) = g_{x}(\beta_i) + \frac{1}{2} \sum_{i=2}^{r_n} \left[ g_{x}(\beta_i) - g_{x}(\beta_{i-1}) \right] + \left| g_{x}(\beta_i) - g_{x}(\beta_{i-1}) \right| 
\]
if \( g_x \) is BV, i.e. for almost all \( x \). But, since \( g(x, \beta^n_r) = g_x(\beta^n_r) \), \( i=1, 2, \cdots, r_n \) is a summable function of \( x \) for \( n=1, 2, 3, \cdots \), it follows that \( g_{\overline{p}}^+(x, y) \) is a measurable, and in fact summable, function of \((x, y) \in Q\). Similarly \( g_{\overline{p}}^+(x, y) \) is a measurable and summable function on \( Q \). Although the form of \( g_p^+ \) and \( g_{\overline{p}}^+ \) is not identical to that of \( \phi_p^+ \) and \( \phi_{\overline{p}}^+ \) in the previous lemma, the only essential distinction is that instead of a \( g_x(\beta_i) \) term, there is a \( g_x(\beta^n_r) \) term appearing, where \( g_x \) is BV. Hence, by the lemma, \( \lim_{n \to \infty} g_p^+(x, y) \) and \( \lim_{n \to \infty} g_{\overline{p}}^+(x, y) \) exist for all \((x, y) \in Q \). Let \( g^+ \) and \( g^- \) be these limits, respectively, then
\[
g^+ \text{ and } g^- \text{ are measurable since each is a limit of a sequence of measurable functions.}
\]

Suppose \( 0 < \alpha < 1 \) and \( 0 \leq \alpha \leq 1 \). Then, there is \( N > 0 \) so that \( n > N \) implies \( P_n \) is such that \( \beta^n_1 \leq \alpha \leq \beta^n_n \). Hence, for \( n > N \), \( g_{\overline{p}}^+(x, \alpha) - g_{\overline{p}}^+(x, \alpha) = \phi_{\overline{p}}^+(g_x, \alpha) - \phi_{\overline{p}}^+(g_x, \alpha) \) if \( g_x \) is BV, and \( g(x, \alpha) \) if \( g_x \) is not BV, and thus, by the lemma, one has that the limit, \( g^+(x, \alpha) - g^-(x, \alpha) \), is either \( g(x, \alpha) \) or \( g(x, \alpha+) \) depending upon whether \( \alpha = \beta^n_i \) for some \( j, n=1, 2, 3, \cdots \) and \( g_x \) is BV. Thus, \( g^+(x, y) - g^-(x, y) = g(x, y) \) at all points \((x, y) \in Q \) such that either \( g_x \) is not BV or \( g_e \) is BV and continuous at \( y \). Since a BV function can be discontinuous at no more than a countable number of points, if \( S \) is the set for which \( g^+ - g^- \) differs from \( g \), \( S \) is measurable since \( g^+ \), \( g^- \) and \( g \) are measurable, \( m(S_n) = 0 \) for all \( x \), where \( S_n = \{ y: (x, y) \in S \} \) and \( m(S) = \int_S m(S_n) dx, \) thus \( m(S) = 0 \). Hence, \( g^+ - g^- \) equals \( g \) almost everywhere on \( Q \) and \( g^+ \) and \( g^- \) are monotone for almost all \( x \in I \).

Also, \( V(g^+ - g^-) \leq V(g_x) \).
It is clear from the definition of $g^+$ and $g^-$, where $\beta \in \mathbb{P}_n$, $n=1, 2, 3, \ldots$, that since $|g(x, y)| \leq |g(x, \beta)| + V(g_\beta)$, it follows that both $|g^+(x, y)|$ and $|g^-(x, y)|$ are bounded by $|g(x, \beta)| + 2V(g_\beta)$ and since $V(g_\beta)$ and $g(x, \beta)$ are both summable on $\Omega$, $g^+$ and $g^-$ are also summable on $\Omega$.

Let $(g^+)^s$ and $(g^-)^s$ be the integral means of $g^+$ and $g^-$, i.e., $(g^+)^s(x, y) = \int_0^s \int_0^s g^+(u, v) \, du \, dv$, $0 \leq x, y \leq 1$, $g^+$ is continued periodically and similarly for $g^-$. It is clear that $(g^+)_s^s$ and $(g^-)_s^s$ are monotone for all $x$ since $g^+_s$ and $g^-_s$ are monotone for almost all $x$ and it is well known that $(g^+)^s$ and $(g^-)^s$ are continuous and converge almost everywhere on $\Omega$ to $g^+$ and $g^-$ as $s$ goes to zero. Thus, $k^+ = \lim sup_{s \to 0} (g^+)^s$ and $k^- = \lim sup_{s \to 0} (g^-)^s$ have the same properties as $g$ relative to $f$; i.e., $k = k^+ - k^-$ is equal almost everywhere to $f$, $k_x$ is BV for almost all $x$, $k^+_x$ and $k^-_x$ are monotone and $V(k_x) \leq V(g_\beta)$ for almost all $x$. Thus, $\int_0^1 V(k_x) \, dx < \infty$.

By exactly the same argument with $h$, the same function $k$ is obtained due to the symmetry of the integral means with respect to $x$ and $y$. Thus, there is a single function, $k$, equal almost everywhere to $f$, for which $k_x$ and $k_y$ are BV for almost all $x$ and $y$ respectively, and

$$\int_0^1 V(k_x) \, dx < \infty \quad \text{and} \quad \int_0^1 V(k_y) \, dy < \infty.$$ 

References


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