

FIXED POINTS AND OPENNESS

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Let $A: X \rightarrow Y$ be a linear mapping¹ of a linear topological space X into a linear topological space Y . Then the image $A(X)$ of X is a linear subset of Y . Thus, if $A(X)$ contains an open subset of Y , there is evidently $A(X) = Y$, i.e. $A: X \rightarrow Y$ is a mapping onto Y .² But, if we know that $A(X) = Y$, we know also that for every $y \in Y$ there exists a solution $x \in X$ of the equation $A(x) = y$. These considerations show the importance of the following

PROBLEM. Let $A: X \rightarrow Y$ be a mapping (not necessarily linear) of a topological space X into a topological space Y . Under what conditions is $A(X)$ open in Y ?

The aim of this paper is to give a particular solution of this problem in the case of mappings $A: X \rightarrow X$ of a Banach space X into itself. It will be shown that the Fixed Point Theorems of Schauder and Brouwer may be applied to find conditions under which the image $A(X)$ of X is open in X . The idea of the following proofs is: Suppose that $A: X \rightarrow X$ is a mapping of X into itself and let $y_0 \in A(X)$. To prove that $A(X)$ contains a spherical region $S(y_0, r_0)$ in X with centre y_0 and radius r_0 , we take an arbitrary point $\bar{y} \in S(y_0, r_0)$ and define the mapping $f(x) = x - \lambda[A(x) - \bar{y}]$. Now, if it is possible to find for every $\bar{y} \in S(y_0, r_0)$ a number $\lambda \neq 0$ (depending on r_0 and \bar{y}) such that $f(x)$ has a fixed point $\bar{x} = f(\bar{x})$, then by $\lambda \neq 0$, we conclude that $\bar{y} = A(\bar{x})$ belongs to $A(X)$. Therefore $S(y_0, r_0) \subset A(X)$. In the following Theorem 1 it will be sufficient to choose $\lambda = 1$. Note that if we know that the image $A(X)$ of X is closed and open in X , then for a connected X we obtain $A(X) = X$. This last idea was used by the author in [1] and [2] to obtain some generalizations of the Fundamental Theorem of Algebra to n -dimensional Euclidean spaces and general Banach spaces.

1. We quote the following

PRINCIPLE OF SCHAUDER. A continuous mapping of a closed, convex set of a Banach space X into a compact subset of this set has a fixed point.³

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¹ By "mapping" we always understand a continuous mapping.

² This remark was made by the referee. It enabled the author to avoid some more complicated considerations.

³ See [3, Theorem 2].

DEFINITION 1. Let $A: X \rightarrow Y$ be a mapping of a metric space X into a metric space Y and let $y_0 \in A(X)$. We say that $A: X \rightarrow Y$ is open at the point y_0 if there exists a spherical region $S(y_0, r_0)$ in Y such that $S(y_0, r_0) \subset A(X)$.

Now let $F: X \rightarrow X$ be a mapping of a Banach space X into itself and denote by

$$M(x_0, r) = \sup_{\|x-x_0\| \leq r} \|F(x) - F(x_0)\|.$$

THEOREM 1. Let $F: X \rightarrow X$ be a completely continuous⁴ mapping of a Banach space X into itself and let $A(x) = x - F(x)$. If, for the point $y_0 \in A(X)$, there exists a point $x_0 \in A^{-1}(y_0)$ such that

$$(1) \quad \inf_{0 < r < \infty} \frac{M(x_0, r)}{r} < 1,$$

then $A: X \rightarrow X$ is open at the point $y_0 \in A(X)$.

PROOF. Let $f(x) = x - [A(x) - \bar{y}]$. We have

$$(2) \quad \begin{aligned} \|f(x) - x_0\| &= \|F(x) + \bar{y} - x_0\| \\ &\leq \|F(x) - F(x_0)\| + \|F(x_0) + \bar{y} - x_0\| \\ &= \|F(x) - F(x_0)\| + \|y_0 - \bar{y}\| \end{aligned}$$

where $y_0 = A(x_0)$. By (1) we can choose r such that for x satisfying $\|x - x_0\| \leq r$ there is $\|F(x) - F(x_0)\| \leq r_1 < r$. Thus, by (2), choosing r_0 such that $0 < r_0 < r - r_1$, we obtain for any $\bar{y} \in S(y_0, r_0)$ that $\|f(x) - x_0\| \leq r_1 + (r - r_1) = r$. This means that $f(x)$ maps the closed spherical region $\bar{S}(x_0, r)$ into itself.⁵ Since $F(x)$ is completely continuous, $f(x) = F(x) + \bar{y}$ is also completely continuous. Thus, by the principle of Schauder, there exists a fixed point $\bar{x} = f(\bar{x})$, i.e. $A(\bar{x}) = \bar{y}$. Hence, every point \bar{y} of $S(y_0, r_0)$ belongs to the image $A(X)$ of X and therefore $A: X \rightarrow X$ is open at the point $y_0 \in A(X)$.

Note that if X is a finite dimensional Euclidean space and $F: X \rightarrow X$ is a mapping of X into itself, then F is completely continuous. Thus from Theorem 1 we obtain the following

COROLLARY 1. Let $F: X \rightarrow X$ be a mapping of the n -dimensional Euclidean space X into itself and let $A(x) = x - F(x)$. If $y_0 \in A(X)$ and if there exists a point $x_0 \in A^{-1}(y_0)$ such that $\inf_{0 < r < \infty} M(x_0, r)/r < 1$ then $A: X \rightarrow X$ is open at the point $y_0 \in A(X)$.

REMARK 1. It is easily seen that it is possible to prove the above corollary in complete analogy with the proof of Theorem 1, by using

⁴ See [4, p. 274].

⁵ \bar{S} denotes the closure of S .

the classical Fixed Point Theorem of Brouwer instead of that of Schauder used in the proof of Theorem 1.

Note also that if $F: X \rightarrow X$ is a locally contractive mapping, i.e. a mapping such that for every point x_0 there exists a spherical region $S(x_0, r_0)$ and a number $\alpha = \alpha(x_0)$, $0 < \alpha < 1$, such that for every point x of $S(x_0, r_0)$ there is $\|F(x) - F(x_0)\| \leq \alpha \|x - x_0\|$, then evidently $\inf_{0 < r < \infty} M(x_0, r)/r \leq \alpha < 1$ for every x_0 and hence by Theorem 1.

COROLLARY 2. *If $F: X \rightarrow X$ is a completely continuous locally contractive mapping of a Banach space X into itself, then $A: X \rightarrow X$ is open at every point $y_0 \in A(X)$ and therefore $A(X)$ is open in X .*

REMARK 2. Taking $F(x) = x$ and X a finite dimensional Euclidean space, we see that the mapping $A(x) = x - x = 0$ is a trivial mapping and $A(X) = 0$. Thus $A(X)$ is not open at the point $0 \in A(X)$. But for $F(x) = x$ there is $M(x_0, r)/r = 1$ for every x_0 and every r and thus the assumption (1) of Theorem 1 does not hold. This shows that the assumption (1) is, in a certain sense, a necessary assumption.

REMARK 3. Let $A: X \rightarrow Y$ be a mapping of a metric space X into a metric space Y such that the condition $\{x_n\}_{n=1,2,\dots}$ does not contain a Cauchy (fundamental) subsequence, $x_n \in X$, implies that $\{A(x_n)\}_{n=1,2,\dots}$ does not contain a Cauchy subsequence. Mappings having this property are called polynomial mappings.⁶ It can be shown⁷ that if $A: X \rightarrow Y$ is a polynomial mapping of a complete metric space X into a connected metric space Y which is open at every point $y \in A(X) \setminus I$ where $I \neq A(X)$ is a set which does not disconnect the space Y , then $A(X) = Y$. Hence, by Theorem 1, we obtain that:

Let $F: X \rightarrow X$ be a completely continuous mapping of a Banach space X into itself and let $A(x) = x - F(x)$. If the condition (1) holds for every point $y_0 \in A(X) \setminus I$, where $I \neq A(X)$ is a set which does not disconnect X and if, in addition, $A: X \rightarrow X$ is a polynomial mapping, then $A(X) = X$.

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4. A. E. Taylor, *Introduction to functional analysis*, New York, John Wiley and Sons, Inc., 1958.

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⁶ See [1, p. 157].

⁷ See [2, p. 1400].