ON THE SUM OF THE ELEMENTS IN THE CHARACTER TABLE OF A FINITE GROUP

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In this note we prove an inequality governing the sum of the elements in the character table of a finite group.

**Theorem.** Let $G$ be a finite group of order $g$. Let $\chi_1, \ldots, \chi_k$ be the absolutely irreducible characters of $G$ and let $G_1, \ldots, G_k$ be representatives for the classes of conjugate elements. Let $h$ be the order of a maximal abelian normal subgroup of $G$ and let

$$s = \sum_{i=1}^{k} \sum_{j=1}^{k} \chi_i(G_j)$$

be the sum of the elements in the character table. Then $s$ is a rational integer and

$$h \leq s \leq g.$$

We have equality $s=g$ if and only if $G$ is nilpotent of class two, and equality $s=h$ if and only if $G$ is abelian.

**Proof.** Let $G \to \psi(G)$ be the permutation representation of $G$ defined by inner automorphisms

$$\psi(G)X = GXG^{-1}, \quad G, X \in G$$

and let $\nu$ be the character of $\psi$. We may write $\nu = \sum_{i=1}^{k} c_i \chi_i$, where the $c_i$ are non-negative rational integers. Since $\nu(G_j)$ is the order of the normalizer of $G_j$ we have $\nu(G_j) = g/k_j$ where $k_j$ is the number of conjugates of $G_j$. It follows from the orthogonality relations that

$$c_i = \frac{1}{g} \sum_{G \in G} \nu(G) \chi_i(G) = \sum_{j} \chi_i(G_j).$$

Let $x_i$ be the degree of $\chi_i$. Since the $c_i$ are non-negative and $\nu$ is a character of degree $g$, it follows that

$$g = \sum_{i} c_i x_i \geq \sum_{i} c_i = s.$$

Thus $s$ is a rational integer and $s \leq g$. Equality holds if and only if $c_i = 0$ whenever $x_i > 1$. Thus equality holds if and only if all the irreducible constituents of $\nu$ have degree one. We shall see that this is the case if and only if the commutator subgroup $G'$ is included in the

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center $Z$ of $G$, and hence if and only if $G$ is nilpotent of class two.
If all the irreducible constituents of $\nu$ are characters $\lambda$ of degree one,
then $\lambda(G) = 1$ for all $G \in \mathcal{G}'$ implies $\nu(G) = g$ for all $G \in \mathcal{G}'$ and thus $\mathcal{G}' \subseteq Z$. Suppose conversely that $\mathcal{G}' \subseteq Z$. Let $x_i$ be a matrix representation of $G$ with character $\chi_i$. Then Schur's Lemma implies $x_i(Z)$ is a multiple of the identity matrix for all $Z \in Z$ and thus $\chi_i(Z) = \omega_i(Z)x_i$ where $\omega_i(Z)$ is a root of unity. Thus for $X, Y \in \mathcal{G}$ we have

$$g = \nu(XX^{-1}Y^{-1}) = \sum_i c_i x_i \omega_i(XX^{-1}Y^{-1}).$$

On the other hand $g = \sum_i c_i x_i$. We use the following familiar property of roots of unity: If $\epsilon_1, \ldots, \epsilon_r$ are roots of unity and $\sum_i \epsilon_i = r$ then $\epsilon_i = 1$ for $i = 1, \ldots, r$. Thus, in our case, $c_i \neq 0$ implies $\omega_i(XX^{-1}Y^{-1}) = 1$ and hence $\chi_i(XX^{-1}Y^{-1}) = x_i$ for all $X, Y \in \mathcal{G}$. From the formula [2]

$$\chi_i(X) \overline{\chi_i(X)} = \frac{x_i}{g} \sum_{Y \in \mathcal{G}} \chi_i(XX^{-1}Y^{-1})$$

we see that $c_i \neq 0$ implies $|\chi_i(X)|^2 = x_i^2$ for all $X \in \mathcal{G}$ and then $g = \sum_{X \in \mathcal{G}} |\chi_i(X)|^2 = g x_i^2$ shows $x = 1$. Thus all the irreducible constituents of $\nu$ have degree one and $s = g$.

To show that $s \geq h$, let $x = \max_i x_i$. Then a theorem of Itô [1] shows $x \leq g/h$. But then, since $c_i \geq 0$ we have

$$s = \sum_i c_i \geq \sum_i c_i \frac{x_i}{x} = \frac{g}{x} \geq h.$$