ON THE SUM OF THE ELEMENTS IN THE CHARACTER TABLE OF A FINITE GROUP

LOUIS SOLOMON

In this note we prove an inequality governing the sum of the elements in the character table of a finite group.

THEOREM. *Let* $\mathfrak{G}$ *be a finite group of order* g. *Let* $\chi_1, \cdots, \chi_k$ *be the absolutely irreducible characters of* $\mathfrak{G}$ *and let* $G_1, \cdots, G_k$ *be representatives for the classes of conjugate elements. Let* h *be the order of a maximal abelian normal subgroup of* $\mathfrak{G}$ *and let*

$$s = \sum_{i=1}^{k} \sum_{j=1}^{k} \chi_i(G_j)$$

be the sum of the elements in the character table. Then s *is a rational integer and*

$$h \leqq s \leqq g.$$

We have equality $s=g$ *if and only if* $\mathfrak{G}$ *is nilpotent of class two, and equality* $s=h$ *if and only if* $\mathfrak{G}$ *is abelian.*

PROOF. Let $G \rightarrow \mathfrak{P}(G)$ be the permutation representation of $\mathfrak{G}$ defined by inner automorphisms

$$\mathfrak{P}(G)X = GXG^{-1} \qquad G, X \in \mathfrak{G}$$

and let ν be the character of $\mathfrak{P}$. We may write $\nu = \sum_{i=1}^{k} c_i\chi_i$ where the c_i are non-negative rational integers. Since $\nu(G_j)$ is the order of the normalizer of G_j we have $\nu(G_j) = g/k_j$ where k_j is the number of conjugates of G_j. It follows from the orthogonality relations that

$$c_i = \frac{1}{g} \sum_{G \in \mathfrak{G}} \nu(G)\chi_i(G) = \sum_j \chi_i(G_j).$$

Let x_i be the degree of χ_i. Since the c_i are non-negative and ν is a character of degree g, it follows that

$$g = \sum_i c_i x_i \geqq \sum_i c_i = s.$$

Thus s is a rational integer and $s \leqq g$. Equality holds if and only if $c_i = 0$ whenever $x_i > 1$. Thus equality holds if and only if all the irreducible constituents of ν have degree one. We shall see that this is the case if and only if the commutator subgroup $\mathfrak{G}'$ is included in the

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center $\mathfrak{Z}$ of $\mathfrak{G}$, and hence if and only if $\mathfrak{G}$ is nilpotent of class two. If all the irreducible constituents of ν are characters λ of degree one, then $\lambda(G)=1$ for all $G\in\mathfrak{G}'$ implies $\nu(G)=g$ for all $G\in\mathfrak{G}'$ and thus $\mathfrak{G}'\subseteq\mathfrak{Z}$. Suppose conversely that $\mathfrak{G}'\subseteq\mathfrak{Z}$. Let $\mathfrak{X}_i$ be a matrix representation of $\mathfrak{G}$ with character χ_i. Then Schur's Lemma implies $\mathfrak{X}_i(Z)$ is a multiple of the identity matrix for all $Z\in\mathfrak{Z}$ and thus $\chi_i(Z)=\omega_i(Z)x_i$ where $\omega_i(Z)$ is a root of unity. Thus for X, $Y\in\mathfrak{G}$ we have

$$g = \nu(XYX^{-1}Y^{-1}) = \sum_i c_i x_i \omega_i(XYX^{-1}Y^{-1}).$$

On the other hand $g=\sum_i c_i x_i$. We use the following familiar property of roots of unity: If $\epsilon_1, \cdots, \epsilon_r$ are roots of unity and $\sum_i \epsilon_i = r$ then $\epsilon_i = 1$ for $i = 1, \cdots, r$. Thus, in our case, $c_i \neq 0$ implies $\omega_i(XYX^{-1}Y^{-1})=1$ and hence $\chi_i(XYX^{-1}Y^{-1})=x_i$ for all X, $Y\in\mathfrak{G}$. From the formula [2]

$$\chi_i(X)\overline{\chi_i(X)} = \frac{x_i}{g}\sum_{Y\in\mathfrak{G}} \chi_i(XYX^{-1}Y^{-1})$$

we see that $c_i\neq 0$ implies $|\chi_i(X)|^2=x_i^2$ for all $X\in\mathfrak{G}$ and then $g=\sum_{X\in\mathfrak{G}}|\chi_i(X)|^2=gx_i^2$ shows $x=1$. Thus all the irreducible constituents of ν have degree one and $s=g$.

To show that $s\geqq h$, let $x=\max_i x_i$. Then a theorem of Itô [1] shows $x\,|\,g/h$. But then, since $c_i\geqq 0$ we have

$$s = \sum_i c_i \geqq \sum_i c_i \frac{x_i}{x} = \frac{g}{x} \geqq h.$$

Clearly if $\mathfrak{G}$ is abelian $s=h=g$. Conversely if $s=h$ then $x_i=x$ whenever $c_i\neq 0$. Thus ν is a linear combination of characters of degree x. But ν is the character of a permutation representation and hence contains the principal character as an irreducible constituent. Thus $x=1$ and $\mathfrak{G}$ is abelian. This completes the proof.

References

1. N. Itô, *On the degrees of irreducible representations of a finite group*, Nagoya Math. J. vol. 3 (1951) pp. 5–6.

2. B. L. van der Waerden, *Modern algebra*, New York, Ungar, 1950, vol. 2, p. 190.

Haverford College