EXTENDING CHARACTERS ON SEMIGROUPS
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W. W. Comfort has proved [1, Theorem 4.2] a theorem on approximating certain semicharacters on commutative semigroups. He used the structure theory established in [2] and expressed doubt as to the necessity of one of his hypotheses, namely core $S(\chi) \neq \Lambda$. His result suggested the following theorem, which tells us when a character on a subsemigroup of a commutative semigroup $G$ can be extended to a character on $G$. Because of its technical nature we will not state Comfort’s theorem but we will state as a corollary to our theorem a result which implies his theorem directly (with the hypothesis core $S(\chi) \neq \Lambda$ dropped).

A bounded complex-valued function $\psi$ on a semigroup $G$ is called a semicharacter of $G$ if $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$. A character $\psi$ is a semicharacter for which $|\psi(x)| = 1$ for all $x \in G$. We note that it follows from the theorem in [3] that any character can be extended to a semicharacter.

**Theorem.** Let $G$ be a commutative semigroup and let $S \subseteq G$ be a subsemigroup. A character $\psi$ on $S$ can be extended to a character on $G$ if and only if $\psi$ satisfies:

\[(*) \quad a, b \in S, x \in G, \text{ and } ax = bx \implies \psi(a) = \psi(b).\]

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References


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Proof. The necessity of (\(*\)) is clear. For the sufficiency, we may suppose that \(G\) has a unit. By Zorn's lemma, it suffices to choose an \(x_0 \in G - S\) and extend \(\psi\) to a character \(\psi_0\) on \(S_0 = \{sx_0^k : s \in S, k \geq 0\}\) that satisfies (\(*\)) on \(S_0\). Three cases must be considered. We omit the details which are similar to those in [3] but we note that in each case the crucial matters to be checked are that \(\psi_0\) is well-defined and that \(\psi_0\) satisfies (\(*\)) on \(S_0\). In Cases 2 and 3, the denials of the previous cases are essential.

Case 1. Suppose there exist \(a_0, b_0 \in S, x_0 \in G - S,\) and \(y_0 \in G\) such that \(a_0x_0y_0 = b_0y_0\). Then extend \(\psi\) to \(\psi_0\) such that \(\psi_0(x_0) = \psi(b_0)/\psi(a_0)\).

Case 2. Suppose Case 1 does not apply but that for some \(x_0 \in G - S\) and some \(k_1 \geq 2,\) we have \(x_0^{k_1} \in S\). Then let \(k_0\) be the least positive integer such that \(x_0^{k_0} \in S\) and extend \(\psi\) to \(\psi_0\) such that \(\psi_0(x_0)\) is any \(k_0\)th root of \(\psi(x_0^{k_1})\).

Case 3. Suppose Cases 1 and 2 do not apply. Then choose \(x_0 \in G - S\) arbitrarily and extend \(\psi\) to \(\psi_0\) so that \(\psi_0(x_0) = 1\).

We now state the corollary implying [1, Theorem 4.2].

Corollary. Let \(\chi\) be a semicharacter on a commutative semigroup such that \(\chi(x) = 0\ or |\chi(x)| = 1\) for all \(x \in G\). Let \(S(\chi) = \{x \in G : |\chi(x)| = 1\}\) and suppose that \(A\) is a subsemigroup of \(G\) such that

(1) \(S(\chi) \subseteq A;\)

(2) \(x \in G, y \in G - A\) imply \(xy \in G - A;\)

(3) \(x, y \in S(\chi), z \in A,\) and \(xs = yz\) imply \(\chi(x) = \chi(y).\)

Then there is a semicharacter \(\psi\) on \(G\) such that \(\{x \in G : |\psi(x)| = 1\} = A\) and \(\chi(x) = \psi(x)\) for \(x \in S(\chi)\).

Proof. Let \(\chi_0\) be \(\chi\) restricted to \(S(\chi)\) and extend to a character \(\psi_0\) on \(A\) using the preceding theorem. Then define \(\psi(x) = \psi_0(x)\) for \(x \in A\) and \(\psi(x) = 0\) for \(x \in G - A\).

Note. The above theorem and the theorem of [3] lead one to ask what conditions are necessary to extend semicharacters that never take the value zero. A natural conjecture would be condition (\(*\)) above and condition (A) of [3]:

\[(A) \quad a, b \in S, x \in G,\) and \(ax = b\) imply \(|\psi(a)| \geq |\psi(b)|.\]

However, consider the following example. Let \(G\) be the commutative semigroup generated by \(\{a_1, a_2, \ldots, b_1, b_2, \ldots, c, d\}\) and satisfying the relations:

\[(**) \quad a_1^1 a_2 a_3 \cdots b_1^{m_1} b_2 b_3 \cdots c^p d^q = a_1^{k_1} a_2^{k_2} \cdots b_1^{m_1'} b_2 b_3 \cdots c^{p'} d^{q'},\]

\[\text{Dr. Comfort suggested this simplification of the author's original example.}\]
whenever \( q > 0, q' > 0, k_n - m_n = k'_n - m'_n \) for all \( n \), \( p - q = q' - p' \), and \( q + \sum_{n=1}^{\infty} m_n = q' + \sum_{n=1}^{\infty} m'_n \). (In the expression \((**\)) all but finitely many of the exponents are zero.) Let \( S \) be the subsemigroup of \( G \) generated by \( \{a_1, a_2, \ldots, c\} \) and define \( \psi \) on \( S \) by

\[
\psi(a_1^{k_1} a_2^{k_2} \cdots a_i^{k_i}) = \prod_{n=1}^{\infty} \left( \frac{1}{n} \right)^{k_n}.
\]

Then

(i) \( \psi \) never takes the value zero on \( S \);
(ii) \( a, b \in S, x \in G, \) and \( ax = bx \) imply \( \psi(a) = \psi(b) \);
(iii) \( a, b \in S, x, y \in G, \) and \( axy = by \) imply \( |\psi(a)| \geq |\psi(b)| \);
(iv) any extension of \( \psi \) to a semicharacter on \( G \) takes on the value zero.

Indeed, if \( \psi_0 \) extends \( \psi \), then \( \psi_0(d) = 0 \) since \( a_n b_n d = c d^2 \) implies that \( |\psi_0(d)| \leq 1/n \) for all \( n \). The example can be considerably simplified if only condition (A) is desired rather than condition (iii).

References


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