CONTINUOUS FUNCTIONS DEFINED ON PRODUCT-SPACES

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1. The results. The most concrete result of this paper is

Theorem 1. Let \( f(x, y) \) be a continuous double-periodic function satisfying \( f(x+1, y) = f(x, y+1) = f(x, y) \). Let \( \alpha, \beta \) be arbitrary. Then there exist \( x, y, \tilde{y} \) having

\[
f(x, y) = f(x, y + \beta) = f(x + \alpha, \tilde{y}) = f(x + \alpha, \tilde{y} + \beta).
\]

Thus \( f \) maps the vertices of a certain parallelogram into a single number. In [1] I proved the theorem in special cases and showed that it has an application to continuous functions on the 3-sphere. In [1] I also showed that the theorem would no longer be true if one would ask for \( y = \tilde{y} \).

More generally, we say a class \( \sigma_k \) of \( k \)-tuples of points in a compact topological space \( S \) has property \( p \), if every real-valued map \( f \) of \( S \) maps all the points of a \( k \)-tuple \( \Sigma \in \sigma_k \) into a single point. Here and throughout the paper, compact means sequentially compact. Thus if \( S \) is the \( n \)-sphere \( S^n \) and \( \sigma_{n+1} \) the class of orthogonal \((n+1)\)-tuples on \( S \), then the Kakutani-Yamabe-Yujobo-Theorem states that \( \sigma_{n+1} \) has property \( p \).

We call the topological product of a line with \( S \) cylinder over \( S \) and denote it by \( C(S) \). Points of \( C(S) \) will be written \((x, X)\), where \( x \) is a real number, \( X \in S \). A continuous curve in \( C(S) \), \( x(t), X(t), -\infty < t < \infty \), will be called a rain over \( S \) if \( x(t) \) tends to \( \pm \infty \) when \( t \) tends to \( \pm \infty \). A roof over \( S \) is a compact set in \( C(S) \) which has a nonempty intersection with every rain over \( S \). A class \( \sigma_k \) of \( k \)-tuples in \( S \) has property \( P \) if to any roof \( R \) over \( S \) there exists a \( k \)-tuple \( \Sigma \in \sigma_k \) and an \( x \) such that

\[
(x, X) \in R \quad \text{for every} \quad X \in \Sigma_R.
\]

Since every real-valued map \( f \) of a compact space \( S \) is associated with the roof \((f(X), X)\), property \( P \) implies \( p \).

Now let \( X_1, \ldots, X_{n+1} \) be an \((n+1)\)-tuple of points on the \( n \)-sphere \( S^n \) whose spherical distances satisfy

\[
d(X_i, X_j) = d(X_1, X_j) \quad (1 \leq i < j \leq n + 1).
\]

Let \( \tau_{n+1} \) be the class of \((n+1)\)-tuples obtained by applying a rotation

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to our particular \( X_1, \ldots, X_{n+1} \). Then the methods of Yamabe-Yujobo [2] show that \( \tau_{n+1} \) has property \( P \).

We say a sequence \( \Sigma_1, \Sigma_2, \ldots \) of \( k \)-tuples is convergent to a \( k \)-tuple \( \Sigma \), if the elements \( X_{i1}, X_{i2}, \ldots, X_{ik} \) of \( \Sigma_i \) and \( X_1, \ldots, X_k \) of \( \Sigma \) can be arranged in such a way that \( \lim X_{ij} = X_j \) (\( j = 1, \ldots, k \)). We call a class \( \sigma_k \) closed if the limit of any convergent sequence of \( k \)-tuples of \( \sigma_k \) is again in \( \sigma_k \).

If \( \sigma_k \) is a class of \( k \)-tuples in \( S \) and \( \tau_l \) a class of \( l \)-tuples in \( T \), then we define \( \sigma \times \tau \) to be the class of the following \( k \cdot l \)-tuples in the topological product \( S \times T \). The \( k \cdot l \)-tuples of \( \sigma \times \tau \) consist of all pairs of the type \((X, Y)\), where \( X \) runs through a \( k \)-tuple \( \Sigma \) of \( \sigma_k \) and, for given \( X \), \( Y \) runs through an \( l \)-tuple \( T_X \) of \( \tau_l \). For example, if \( S = T \) is the space of real numbers modulo 1 and \( \sigma_2(\alpha) \) the class of pairs \((x, x')\) having \( x - x' = \alpha \), then \( \sigma_4(\alpha, \beta) = \sigma_2(\alpha) \times \sigma_2(\beta) \) consists of quadruples \((x, y), (x, y + \beta), (x + \alpha, y), (x + \alpha, y + \beta)\).

**Theorem 2.** Assume \( \sigma_k \) has property \( P \) in \( S \), \( \tau_l \) has property \( P \) in \( T \) and \( \tau_l \) is closed. Then \( \sigma \times \tau \) has property \( P \) in \( S \times T \).

It appears to be difficult to generalize our results to maps \( f \) into \( \mathbb{R}^n \) and to prove the following generalization of the Borsuk-Ulam Theorem: Let \( X \rightarrow -X \) be the antipodal map in \( S^n \) and let \( f \) be a map of \( S^n \times S^n \) into \( \mathbb{R}^n \). Then there exist \( X, Y, \overline{Y} \) in \( S^n \) having \( f(X, Y) = f(X, -Y) = f(-X, Y) = f(-X, -Y) \).

2. The proofs.

**Lemma 1.** Assume \( R \) is a roof over \( S \times T \) and let \( x(t), X(t) \) be a rain \( N \) over \( S \). Then the set \( G(N) \) of points \((t, Y)\) of \( C(T) \) where

\[
(x(t), X(t), Y) \in R
\]

forms a roof over \( T \).

**Proof.** If \((t_n, Y_n)\) is a sequence in \( G(N) \), then \((x(t_n), X(t_n), Y_n) \in R \) has a subsequence convergent to some \((x, X, Y) \in R \). For this subsequence \( x(t_n) \), and therefore \( t_n \), is bounded, and \( t_n \) will have a limit-point \( t_0 \) where \( x = x(t_0) \), \( X = X(t_0) \). Thus \((t_0, Y)\) will be a limit-point of \((t_n, Y_n)\) in \( G(N) \), and \( G(N) \) is compact.

Thus if \( G(N) \) were not a roof, there would exist a rain \( t(s), Y(s) \) over \( T \), having no point in \( G \). Then \( x(t(s)), X(t(s)), Y(s) \) would be a rain over \( S \times T \) with no point in \( R \).

**Lemma 2.** Let \( R \) be a roof over \( S \times T \) and assume \( \tau_l \) of \( T \) is closed and has property \( P \). Then the set \( H \) of points \((x, X)\) in \( C(S) \) such that for suitable \( T(x, X) \in \tau \)
(x, X, Y) ∈ R for every Y ∈ T

is a roof over S.

Proof. By $R^{(i)}$ denote the set of points $(x, X, Y_1, \ldots, Y_l)$ of $C(S \times T \times \cdots \times T)$ such that $(x, X, Y_j) \in R$ ($j=1, \ldots, l$) and $Y_1, \ldots, Y_l$ is an l-tuple of $\tau_l$. It follows from the compactness of $R$ and the closedness of $\tau_l$ that $R^{(i)}$ is compact. $(x, X)$ is in $H$ if and only if there exist $Y_1, \ldots, Y_l$ with $(x, X, Y_1, \ldots, Y_l) \in R^{(i)}$. Therefore $H$ is compact.

Now let $N$ be a rain over $S$. Then $G(N)$ is a roof over $T$ and there exists some $t$ and some $T \in \tau$ such that $(t, Y) \in G$ for every $Y \in T$. Then $(x(t), X(t), Y) \in R$ for every $Y \in T$ and $N$ has a common point with $H$.

Proof of Theorem 2. Assume the hypotheses of the theorem to be satisfied. Construct $H$ as in Lemma 2. By the property of $\sigma$, there exists a $\Sigma \subseteq \sigma$ and some $x$ such that

$$(x, X) \in H \quad \text{for every} \quad X \in \Sigma.$$ Then $(x, X, Y) \in R$ for $X \in \Sigma$, $Y \in T_X$ and Theorem 2 is proved.

Proof of Theorem 1. If $S$ is the space of real numbers modulo 1 and $\sigma_2(\alpha)$ is defined as before, then $\sigma_2(\alpha)$ has property $P$. This is the one-dimensional case of the generalized Yamabe-Yujobo Theorem. Furthermore, $\sigma_2$ is closed. Theorem 1 is a consequence of these facts and Theorem 2.

References


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