NOTE ON LEBESGUE'S CONSTANTS

L. CARLITZ[1]

1. Lebesgue's constants are defined by

$$(1)\qquad L_n = \frac{2}{\pi}\int_0^{\pi/2}\frac{|\sin(2n+1)t|}{\sin t}\,dt = \frac{1}{\pi}\int_0^{\pi}\frac{|\sin(2n+1)t|}{\sin t}\,dt.$$

Fejér [2, p. 103] has proved that

$$(2)\qquad L_n = \frac{1}{2n+1} + \frac{2}{\pi}\sum_{r=1}^{n}\frac{1}{r}\tan\frac{r\pi}{2n+1}\cdot$$

For other references see Hardy [4].

It may be of interest to point out that the equivalence of (1) and (2) can be proved in a very elementary way using only the following formulas:

$$(3)\qquad 1 + 2\sum_{r=1}^{n}\cos 2rx = \frac{\sin(2n+1)x}{\sin x},$$

$$(4)\qquad \sum_{r=1}^{n}(-1)^r\sin rx = (-1)^n\frac{\sin\left(n+\frac{1}{2}\right)x - \sin\frac{1}{2}x}{2\cos\frac{1}{2}x}\cdot$$

We have from (1)

$$\begin{aligned}
\pi L_n &= \int_0^{\pi}\frac{|\sin(2n+1)t|}{\sin t}\,dt = \sum_{r=0}^{2n}(-1)^r\int_{r\pi/(2n+1)}^{(r+1)\pi/(2n+1)}\frac{\sin(2n+1)t}{\sin t}\,dt\\
&= \sum_{r=0}^{2n}(-1)^r\int_{r/(2n+1)}^{(r+1)\pi/(2n+1)}\left\{1 + 2\sum_{s=1}^{n}\cos 2st\right\}dt\\
&= \sum_{r=0}^{2n}(-1)^r\left\{\frac{\pi}{2n+1} + \sum_{s=1}^{n}\left(\sin\frac{2(r+1)s\pi}{2n+1} - \sin\frac{2rs\pi}{2n+1}\right)\right\}\\
&= \frac{\pi}{2n+1} + \sum_{s=1}^{n}\frac{1}{s}\sum_{r=0}^{2n}(-1)^r\left(\sin\frac{2(r+1)s\pi}{2n+1} - \sin\frac{2rs\pi}{2n+1}\right)\\
&= \frac{\pi}{2n+1} + 2\sum_{s=1}^{n}\frac{1}{s}\sum_{r=1}^{2n}(-1)^{r-1}\sin\frac{2rs\pi}{2n+1}\cdot
\end{aligned}$$

Received by the editors December 15, 1960.

[1] Supported in part by National Science Foundation grant G 9425.

But by (4)

$$\sum_{r=1}^{2n}(-1)^{r-1}\sin\frac{2rs\pi}{2n+1}=\frac{\sin\dfrac{s\pi}{2n+1}-\sin\dfrac{(4n+1)s\pi}{2n+1}}{2\cos\dfrac{s\pi}{2n+1}}$$

$$=\frac{2\sin\dfrac{s\pi}{2n+1}}{2\cos\dfrac{s\pi}{2n+1}}=\tan\frac{s\pi}{2n+1},$$

so that

$$\pi L_n=\frac{\pi}{2n+1}+2\sum_{n=1}^{n}\frac{1}{s}\tan\frac{s\pi}{2n+1}\cdot$$

This proves (2).

2. Consider the sum

$$S_n=\sum_{k=0}^{(2n+1)t-1}\frac{\left|\sin\dfrac{k\pi}{t}\right|}{\sin\dfrac{k\pi}{(2n+1)t}},\tag{5}$$

where t is an arbitrary integer $\geqq 1$; for $k=0$ the summand is taken equal to $2n+1$. Then we have

$$S_n=\sum_{r=0}^{2n}(-1)^r\sum_{k=rt}^{(r+1)t-1}\frac{\sin\dfrac{k\pi}{t}}{\sin\dfrac{k\pi}{(2n+1)t}}$$

$$=\sum_{r=0}^{2n}(-1)^r\sum_{k=rt}^{(r+1)t-1}\left\{1+2\sum_{s=1}^{n}\cos\frac{2sk\pi}{(2n+1)t}\right\}$$

$$=t+2\sum_{s=1}^{n}\sum_{r=0}^{2n}(-1)^r\sum_{k=rt}^{(r+1)t-1}\cos\frac{2sk\pi}{(2n+1)t}\cdot$$

Replacing the cosines by exponentials we easily verify that

$$\sum_{r=0}^{2n} (-1)^r \sum_{k=rt}^{(r+1)t-1} \cos \frac{2sk\pi}{(2n+1)t} = \frac{\tan \dfrac{s\pi}{2n+1}}{\tan \dfrac{s\pi}{(2n+1)t}} .$$

We have therefore

$$S_n = t + 2 \sum_{s=1}^{n} \frac{\tan \dfrac{s\pi}{2n+1}}{\tan \dfrac{s\pi}{(2n+1)t}} . \tag{6}$$

If we divide both sides of (6) by $(2n+1)t$ and let $t \to \infty$, (6) reduces to (2). Thus S_n may be thought of as a finite analog of L_n (compare [**1**; **3**] for finite analogs of other integral formulas).

3. If we put

$$L_{n,k} = \frac{2}{\pi} \int_0^{\pi/2} \frac{|\sin(2n+1)kt|}{\sin kt} dt, \tag{7}$$

where k is an arbitrary positive integer, then exactly as above we find that

$$\pi L_{n,k} = \frac{\pi}{2n+1} + \frac{2}{k} \sum_{s=1}^{n} \frac{1}{s} \sum_{r=1}^{2n} (-1)^{r-1} \sin \frac{2krs\pi}{2n+1}$$
$$= \frac{\pi}{2n+1} + \frac{2}{k} \sum_{s=1}^{n} \frac{1}{s} \tan \frac{ks\pi}{2n+1},$$

so that

$$L_{n,k} = \frac{1}{2n+1} + \frac{2}{\pi} \sum_{s=1}^{n} \frac{1}{ks} \tan \frac{ks\pi}{2n+1} . \tag{8}$$

If k and $2n+1$ are relatively prime, this formula can also be written in the following way:

$$L_{n,k} = \frac{1}{2n+1} + \frac{2}{k\pi} \sum_{s=1}^{n} \frac{1}{\{s/k\}} \tan \frac{s\pi}{2n+1}, \tag{9}$$

where $\{s/k\}$ is the integer determined by

$$k\{s/k\} \equiv s \pmod{2n+1}, \qquad -n \leqq \{s/k\} \leqq n.$$

In particular for $k=n$, (9) reduces to

$$L_{n,n} = \frac{1}{2n+1} \tag{10}$$
$$+ \frac{2}{n\pi}\left\{-\sum_{2s\leq n}\frac{1}{2s}\tan\frac{s\pi}{2n+1} + \sum_{2s>n}\frac{1}{2n+1-2s}\tan\frac{s\pi}{2n+1}\right\}.$$

Corresponding to (6) we have the following finite analog of (9):

$$\sum_{s=0}^{(2n+1)t-1} \frac{\left|\sin\frac{ks\pi}{t}\right|}{\sin\frac{ks\pi}{(2n+1)t}} = t + 2\sum_{s=1}^{n} \frac{\tan\frac{ks\pi}{2n+1}}{\tan\frac{ks\pi}{(2n+1)t}}. \tag{11}$$

References

1. W. A. Al-Salam and L. Carlitz, *Some finite summation formulas for the classical orthogonal polynomials*, Rend. Mat. e Appl. vol. 16 (1957) pp. 74–95.

2. L. Fejér, *Sur les singularités de la série de Fourier des fonctions continues*, Ann. Sci. École Norm. Sup. vol. 28 (1911) pp. 63–104.

3. I. J. Good, *A new finite series for Legendre polynomials*, Proc. Cambridge Philos. Soc. vol. 51 (1955) pp. 385–388.

4. G. H. Hardy, *Note on Lebesgue's constants in the theory of Fourier series*, J. London Math. Soc. vol. 17 (1942) pp. 4–13.

Duke University