IDENTITIES IN FINITE PARTITION LATTICES

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In his discussion of the problem of imbedding a finite lattice into a finite partition lattice, Birkhoff [1] speculates that there are no non-trivial identities satisfied in every finite partition lattice. We give a simple proof of this conjecture based upon Whitman's Theorem [4]. Since a partition lattice P is a complete, meet-continuous lattice in which every element is a join of points [3], P is isomorphic to the lattice of ideals of its sublattice P' of finite-dimensional elements. We prove Birkhoff's conjecture by focusing attention on P' rather than on P directly.

We denote the join and meet operations by $+$ and $\cdot$. A lattice polynomial form will be denoted by $f(x_1, \cdots, x_1, \cdots, x_n, \cdots, x_n)$, where we repeat a variable m times if it appears m times in the form.

LEMMA. *If an identity is valid in a lattice L, then it is valid in the lattice L' of all ideals of L.*

PROOF. Let the identity be

$$\begin{aligned} &f(x_1, \cdots, x_1, \cdots, x_n, \cdots, x_n) \\ &= g(x_1, \cdots, x_1, \cdots, x_n, \cdots, x_n). \end{aligned}$$

If $t \in f(X_1, \cdots, X_1, \cdots, X_n, \cdots, X_n)$, then

$$t \leqq f(x_{11}, \cdots, x_{1i}, \cdots, x_{n1}, \cdots, x_{nj})$$

where $x_{pq} \in X_p$, the X_p being ideals. Obviously

$$\begin{aligned} t &\leqq f(x_1', \cdots, x_1', \cdots, x_n', \cdots, x_n') \\ &= g(x_1', \cdots, x_1', \cdots, x_n', \cdots, x_n') \end{aligned}$$

where $x_i' = \sum_p x_{ip}$.

Thus $t \in g(X_1, \cdots, X_1, \cdots, X_n, \cdots, X_n)$ since the X_p are ideals. It therefore follows that $f(X_1, \cdots, X_1, \cdots, X_n, \cdots, X_n) \subseteq g(X_1, \cdots, X_1, \cdots, X_n, \cdots, X_n)$, and by symmetry the identity $f = g$ is valid in L'.

REMARK. This lemma is an exercise (unanswered) in [2, p. 80].

THEOREM. *Any identity valid in every finite partition lattice is trivial, i.e. is valid in every lattice.*

PROOF. Let us suppose that the identity $f = g$ is valid in every finite

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partition lattice. Let P be any partition lattice, and let P' be its sublattice of finite-dimensional elements. The join of any n elements in P' lies in an interval sublattice $[0, a]$ which is isomorphic to a finite partition lattice. (Take the union of all the sets of the n partitions which are nonsingletons.) Thus if the identity $f=g$ is valid in every finite partition lattice, it is valid in P'. In view of the lemma and the fact that P is isomorphic to the lattice of ideals in P', the identity $f=g$ is valid in P. By Whitman's Theorem every lattice is a sublattice of some partition lattice, and therefore the identity $f=g$ is valid in every lattice.

COROLLARY. *There are no nontrivial identities valid in any infinite partition lattice.*

COROLLARY. *For every nontrivial identity $f=g$, there exists a finite lattice in which it is invalid.*

The last corollary suggests the following question: If an identity is valid in every finite sublattice of a lattice L, is it valid in L? This is certainly the case for the modular and distributive identities.

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