MATHEMATICAL PEARLS

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

A RECURRENCE FORMULA FOR \( f(2n) \)

L. CARLITZ

H. T. Kuo [2] has proved the formula

\[
C_{2n} = \frac{\pi (2\pi)^{2n-1}}{4((n - 1)!)^2(2n - 1)} + \frac{1}{(n - 1)!} \sum_{k=0}^{[n/2]} \frac{C_{2k}(2\pi)^{2n-2k}}{(n - 2k)!(2n - 2k)} \\
+ \frac{1}{\pi} \sum_{k=0}^{[n/2]} \sum_{j=0}^{[n/2]} (-1)^{k-j} \frac{C_{2k}C_{2j}(2\pi)^{2n-2k-2j+1}}{(n - 2k)!(n - 2j)!(2n - 2k - 2j + 1)}
\]

where \( n \geq 1 \) and \( C_0 = -1/2, C_{2k} = \xi(2k) \) for \( k = 1, 2, 3, \ldots \).

Since

\[ \xi(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}, \]

where \( B_{2k} \) is a Bernoulli number in the even suffix notation, (1) implies a rather complicated recursion formula for the Bernoulli numbers. It may be of interest to point out that a somewhat simpler and slightly more general formula of this kind can be obtained very rapidly by making use of the identity ([1; 3, p. 75])

\[
B_m(x)B_n(x) = \sum_r \left( \begin{array}{c} m \\ 2r \end{array} \right) n + \left( \begin{array}{c} n \\ 2r \end{array} \right) m \frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r} \\
+ (-1)^{m-1} \frac{m!n!}{(m+n)!} B_{m+n} \quad (m + n \geq 2),
\]

where

\[ B_n(x) = \sum_{k=0}^{n} \left( \begin{array}{c} k \\ n \end{array} \right) B_k x^{n-k}. \]

Integrating (2) from 0 to 1 and making use of

\[
\int_0^1 B_{2k}(x)dx = \frac{B_{2k+1}(1) - B_{2k+1}}{2k + 1} = 0 \quad (k \geq 1),
\]

we get

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991
\[
\sum_{j=0}^{m} \sum_{k=0}^{n} \binom{m}{j} \binom{n}{k} \frac{B_j B_k}{m + n - j - k + 1} = (-1)^{m-1} \frac{m!n!}{(m+n)!} B_{m+n} \quad (m + n \text{ even, } mn > 0).
\]

REFERENCES


Duke University

MINKOWSKI'S THEOREM ON NONHOMOGENEOUS APPROXIMATION

IVAN NIVEN

For \( \theta \) irrational, let \( \gamma = \theta a + b \) has no solution in integers \( a \) and \( b \). We give a short proof of Minkowski’s classic result that there are infinitely many pairs of integers satisfying \( F < 1/4 \), where \( F = F(\theta, \gamma, x, y) = |x| \cdot |\theta x + y + \gamma| \). First we prove that given any real numbers \( \alpha \) and \( \beta \) there exists an integer \( u \) such that \( |u - \beta| < 1 \) and such that at least one of the following holds:

(A) \[ |u - \alpha| \cdot |u - \beta| \leq 1/4; \quad |u - \alpha| \cdot |u - \beta| \leq |\beta - \alpha|/2. \]

If \( \beta \) is an integer, set \( u = \beta \). Otherwise define the integer \( n \) by \( n < \beta < n+1 \). If \( n \leq \alpha \leq n+1 \) then \( |n - \alpha| \cdot |n+1 - \alpha| \leq 1/4 \) and similarly for \( \beta \), and so

\[ |n - \alpha| \cdot |n - \beta| \cdot |n + 1 - \alpha| \cdot |n + 1 - \beta| \leq 1/16. \]

Hence \( u = n \) or \( u = n+1 \) gives inequality (A1). The cases \( n > \alpha \) and \( \alpha > n+1 \) are symmetric, and we treat \( n > \alpha \). We note that

\[ 2(n - \alpha)^{1/2}(n + 1 - \beta)^{1/2} + (\beta - n)(n + 1 - \alpha) = \beta - a \]

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