MATHEMATICAL PEARLS

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

A RECURRENCE FORMULA FOR $\zeta(2n)$

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H. T. Kuo [2] has proved the formula

$$
\begin{aligned}
C_{2n} = {} & \frac{\pi(2\pi)^{2n-1}}{4((n-1)!)^2(2n-1)} + \frac{1}{(n-1)!}\sum_{k=0}^{[n/2]} \frac{C_{2k}(2\pi)^{2n-2k}}{(n-2k)!(2n-2k)} \\
& + \frac{1}{\pi}\sum_{k=0}^{[n/2]}\sum_{j=0}^{[n/2]} (-1)^{k-j} \frac{C_{2k}C_{2j}(2\pi)^{2n-2k-2j+1}}{(n-2k)!(n-2j)!(2n-2k-2j+1)}
\end{aligned} \tag{1}
$$

where $n \geqq 1$ and $C_0 = -1/2$, $C_{2k} = \zeta(2k)$ for $k = 1, 2, 3, \cdots$.

Since

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k},$$

where B_{2k} is a Bernoulli number in the even suffix notation, (1) implies a rather complicated recursion formula for the Bernoulli numbers. It may be of interest to point out that a somewhat simpler and slightly more general formula of this kind can be obtained very rapidly by making use of the identity ([1; 3, p. 75])

$$
\begin{aligned}
B_m(x)B_n(x) = {} & \sum_r \left\{\binom{m}{2r} n + \binom{n}{2r} m\right\} \frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r} \\
& + (-1)^{m-1} \frac{m!n!}{(m+n)!} B_{m+n} \qquad (m+n \geqq 2),
\end{aligned} \tag{2}
$$

where

$$B_n(x) = \sum_{k=0}^{n} \binom{k}{n} B_k x^{n-k}.$$

Integrating (2) from 0 to 1 and making use of

$$\int_0^1 B_{2k}(x)dx = \frac{B_{2k+1}(1) - B_{2k+1}}{2k+1} = 0 \qquad (k \geqq 1),$$

we get

Received by the editors March 6, 1961.

[1] Supported in part by National Science Foundation, grant G-9425.

$$\sum_{j=0}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{n}{k}\frac{B_jB_k}{m+n-j-k+1} = (-1)^{m-1}\frac{m!n!}{(m+n)!}B_{m+n} \qquad (m+n \text{ even}, mn>0). \tag{3}$$

References

1. L. Carlitz, *Note on the integral of the product of several Bernoulli polynomials*, J. London Math. Soc. vol. 34 (1959) pp. 361–363.

2. Huan-Ting Kuo, *A recurrence formula for* $\zeta(2n)$, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 573–574.

3. N. Nielsen, *Traité élémentaire des nombres de Bernoulli*, Paris, Gauthier-Villars, 1923.

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MINKOWSKI'S THEOREM ON NONHOMOGENEOUS APPROXIMATION

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For θ irrational, let γ be any real number such that $\gamma=\theta a+b$ has no solution in integers a and b. We give a short proof of Minkowski's classic result that there are infinitely many pairs of integers satisfying $F<1/4$, where $F=F(\theta, \gamma, x, y)=|x|\cdot|\theta x+y+\gamma|$. First we prove that given any real numbers α and β there exists an integer u such that $|u-\beta|<1$ and such that at least one of the following holds:

(A) $|u-\alpha|\cdot|u-\beta| \leqq 1/4$; $|u-\alpha|\cdot|u-\beta| \leqq |\beta-\alpha|/2$.

If β is an integer, set $u=\beta$. Otherwise define the integer n by $n<\beta<n+1$. If $n\leqq\alpha\leqq n+1$ then $|n-\alpha|\cdot|n+1-\alpha|\leqq 1/4$ and similarly for β, and so

$$|n-\alpha|\cdot|n-\beta|\cdot|n+1-\alpha|\cdot|n+1-\beta| \leqq 1/16.$$

Hence $u=n$ or $u=n+1$ gives inequality (A_1). The cases $n>\alpha$ and $\alpha>n+1$ are symmetric, and we treat $n>\alpha$. We note that

$$2(n-\alpha)^{1/2}(n+1-\beta)^{1/2}(\beta-n)^{1/2}(n+1-\alpha)^{1/2} \leqq (n-\alpha)(n+1-\beta)+(\beta-n)(n+1-\alpha)=\beta-a$$

Received by the editors April 14, 1961.

[1] Supported in part by the Office of Naval Research.