TOROIDAL ALGEBRAIC GROUPS[1]

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Many of the more striking elementary properties of abelian varieties generalize to other kinds of algebraic groups, e.g. tori, i.e. direct products of multiplicative groups G_m, and in fact to extensions of tori by abelian varieties. This seems to have been more or less known for some time, but explicit statements are hard to find, a lack we attempt to remedy here. We may mention specifically a recent paper of S. Arima [1], reading between the lines of which might lead one in this direction. In what follows, the better-known structural facts on algebraic groups are used freely; most of these are in [2] and [4], while the relevant facts on abelian varieties can be found in either [7] or [3].

PROPOSITION. *The following properties of a connected algebraic group G are equivalent*:

(1) *The maximal connected linear algebraic subgroup of G is a torus.*

(2) *G contains no algebraic subgroup that is isomorphic to the additive group G_a.*

(3) *For any connected algebraic subgroup H of G, the points of H of finite order prime to the field characteristic are dense in H.*

PROOF. The equivalence of (1) and (2) is a consequence of the fact that a connected linear algebraic group is a torus if and only if it contains no G_a (cf. [2]). If (1) or (2) holds, clearly any connected algebraic subgroup of G is of the same type. If T is the maximal connected linear subgroup of G then the points of finite order prime to the field characteristic of the abelian variety G/T are dense in G/T, and similarly for the torus T. Thus we get (3). Conversely, (3) trivially gives (2).

A group G such as in the proposition we shall call *toroidal.* Such a group is necessarily commutative and has only a finite number of elements of any given finite order n. As a matter of fact, if n is prime to the field characteristic the elements of order dividing n form a subgroup of G that is the direct product of a certain number of cyclic groups of order n, this number being $\dim T+2\dim G/T$ if T is the

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maximal torus of G; this is immediate from the corresponding facts for tori and abelian varieties, and is the basic tool of Arima [**1**], who takes n to be a variable power of a fixed prime l ($\neq$characteristic) and so gets l-adic matricial representations for homomorphisms between toroidal groups.

COROLLARY. *If the toroidal group G is a normal algebraic subgroup of the connected algebraic group Γ, then G is contained in the center of Γ. Any algebraic group that is isogenous to a toroidal group, or is a connected algebraic subgroup of a toroidal group, or a homomorphic image of a toroidal group, or an extension of one toroidal group by another, is also toroidal.*

The first fact follows from consideration of points of finite order, the rest from the corresponding facts for abelian varieties and tori.

LEMMA. *Let V be a variety, A an abelian variety. Then the rational maps ϕ: $V \rightarrow A$ form, modulo constant maps, a free abelian group with a finite number of generators.*

That there is no torsion in this group is clear, so finite generation is all that must be shown. This may be done by replacing V by its albanese variety, noting that any rational map from an abelian variety B into A is (modulo a translation) a homomorphism, and using the knowledge that $\operatorname{Hom}(B, A)$ is a finite Z-module. However we wish to make two remarks: First, the theory of the albanese variety is not necessary for the proof, for V may be replaced by a generic curve on it (cf. [**3**, pp. 41–42]), hence by a jacobian variety. Second, a very easy proof may be given in the classical case. For this, note that since a rational map ϕ: $V \rightarrow A$ is defined at each simple point of V, we may restrict our attention to such maps ϕ that are everywhere defined. It then suffices to show that if ϕ induces the zero map on the homology group $H_1(V, Z)$ then ϕ is constant. In this case, uniformizing A, we are reduced to showing that any bounded holomorphic function on V is constant, which reduces to the easy case $\dim V = 1$.

It is easy to see that for each of the following Theorems 1–3, the given property actually characterizes toroidal groups in the class of all connected algebraic groups.

THEOREM 1. *Let V be a variety, G a toroidal algebraic group. Then the everywhere defined rational maps ϕ: $V \rightarrow G$ form, modulo constant maps, a free abelian group with a finite number of generators.*

The lemma reduces us to the case where G is a torus, hence to the case $G = G_m$. But this latter case is known [**5**, lemma to Proposition

3]. (Outline of proof for the case $G=G_m$: We may take V to be normal and affine, hence the affine part of a normal projective variety $\overline{V}$. But an everywhere defined nowhere zero function on V is determined, up to a constant factor, by its orders on the various components of $\overline{V}-V$.)

THEOREM 2. *Let V, W be varieties, G a toroidal algebraic group, ϕ: $V\times W\to G$ an everywhere defined rational map. Then there exist everywhere defined rational maps ϕ_1: $V\to G$, ϕ_2: $W\to G$ such that, for any $(v, w)\in V\times W$, $\phi(v, w)=\phi_1(v)+\phi_2(w)$.*

Letting $\psi_1, \cdots, \psi_r$: $V\to G$ be a set of generators, modulo constants, for the group of all everywhere defined rational maps ψ: $V\to G$, letting k be a field of definition for V, W, G, ϕ, $\psi_1, \cdots, \psi_r$, and letting P be a generic point of W over k, there exist integers $n_1, \cdots, n_r$ such that

$$\phi(v, P) = n_1\psi_1(v) + \cdots + n_r\psi_r(v) + f_P,$$

where $f_P\in G$ is a constant point, rational over $k(P)$. All we have to do now is let ϕ_2: $W\to G$ be the rational map, defined over k, such that $\phi_2(P)=f_P$.

THEOREM 3. *Let ϕ: $\Gamma\to G$ be an everywhere defined rational map from a connected algebraic group Γ into a toroidal algebraic group G, with $\phi(e)=0$. Then ϕ is a homomorphism.*

An algebraic group of the form $G=AT$, where A, T are algebraic subgroups of G, A being abelian and T a torus, is clearly toroidal, but not all toroidal groups are of this form. For an example, start with a nonsingular curve C and $r>1$ distinct points $P_1, \cdots, P_r$ of C. Then there is a projective model $C_{\mathfrak{m}}$ of C which is nonsingular except for one point, at which the local ring consists precisely of all functions on C which are defined and take on equal values at $P_1, \cdots, P_r$ ($\mathfrak{m}$ denotes the divisor $P_1+\cdots+P_r$). The generalized jacobian $J_{\mathfrak{m}}$ of $C_{\mathfrak{m}}$ is then toroidal, its maximal torus having dimension $r-1$. If $J_{\mathfrak{m}}$ were of the form AT it would possess a nontrivial homomorphism into a torus, hence into G_m, and the restriction of such a function to the canonical image of C would be a nonconstant rational function on C the carrier of whose divisor is contained in $|\mathfrak{m}|=P_1\cup\cdots\cup P_r$, and we know that such a function cannot exist for C of genus >0 and $P_1, \cdots, P_r$ chosen properly. However Arima has shown [1] that any toroidal group that is defined over a finite field is actually of the form AT; in fact he has the following slightly stronger result, for which we offer a different proof.

THEOREM 4. *If G is a connected algebraic group that is defined over a finite field and A is its maximal abelian subvariety and L its maximal connected linear algebraic subgroup, then $G=AL$.*

The smallest normal algebraic subgroup D of G such that G/D is linear is, according to the general structure theory [**4**], connected and commutative, contains only a finite number of elements of any given finite order (in particular, order equal to the field characteristic), and is such that $G=DL$, L being as above. Since D is toroidal, we may assume to begin with that G is toroidal. G is generated by its curves through 0 that are defined over finite fields, hence may be assumed to be generated by one such curve, so that, according to [**6**], it is a homomorphic image of a generalized jacobian variety. As a matter of fact, since G is toroidal this generalized jacobian may be taken to be of the type $J_{\mathfrak{m}}$ of the previous paragraph; this follows from [**6**, p. 84], or by noting that since the unipotent part of this generalized jacobian must map into 0 in G the generalized jacobian may be replaced by a quotient, which is now another generalized jacobian, of the type $J_{\mathfrak{m}}$. Hence we may suppose that $G=J_{\mathfrak{m}}$, where $\mathfrak{m}=P_1+ \cdots +P_r$ is a divisor on a curve C, C and each P_i being defined over a finite field. For each $i=1, \cdots, r-1$, the divisor P_i-P_r has an image on the ordinary jacobian variety of C that is contained in the subgroup of points that are rational over a certain finite field, so that P_i-P_r is of finite order in the divisor class group, i.e. there is a rational function f_i on C such that $f_i(P_i)=0, f_i(P_r)=\infty$ and f_i is elsewhere finite and nonzero. The rational map $\psi: C\rightarrow(G_m)^{r-1}$ that is given by $\psi(p)=(f_1(p), \cdots, f_{r-1}(p))$ is defined at each point of $C-|\mathfrak{m}|$. Using the universal mapping property of generalized jacobians, as above, we get a rational homomorphism $\tau: J_{\mathfrak{m}}\rightarrow(G_m)^{r-1}$ such that $\psi(p)=\tau(\phi(p))+\alpha$, $\phi: C\rightarrow J_{\mathfrak{m}}$ being the canonical map and α being a constant. Altering each f_i by a nonzero constant factor, we can assume $\alpha=0$, so $\psi=\tau\phi$. τ must be surjective, for otherwise there would be integers $n_1, \cdots, n_{r-1}$, not all zero, such that $f_1^{n_1} \cdots f_{r-1}^{n_{r-1}}=1$, which is impossible. Noting that $J_{\mathfrak{m}}$ is an extension of a torus of dimension $r-1$ by the ordinary jacobian of C, we see that the component of the identity of the kernel of τ is an abelian variety of the correct dimension.

In this paper we have refrained from considering problems involving fields of definition. There are a number of such problems of interest, for example the theory of the K/k-image and K/k-trace of a toroidal group defined over an extension field K of k (oral remark of S. Lang), but the tools for handling these are at hand.

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EXTENDING CHARACTERS ON SEMIGROUPS

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W. W. Comfort has proved [1, Theorem 4.2] a theorem on approximating certain semicharacters on commutative semigroups. He used the structure theory established in [2] and expressed doubt as to the necessity of one of his hypotheses, namely core $S(\chi)\neq\Lambda$. His result suggested the following theorem, which tells us when a character on a subsemigroup of a commutative semigroup G can be extended to a character on G. Because of its technical nature we will not state Comfort's theorem but we will state as a corollary to our theorem a result which implies his theorem directly (with the hypothesis core $S(\chi)\neq\Lambda$ dropped).

A bounded complex-valued function ψ on a semigroup G is called a *semicharacter* of G if $\psi(x)\neq 0$ for some $x\in G$ and $\psi(xy)=\psi(x)\psi(y)$ for all $x, y\in G$. A *character* ψ is a semicharacter for which $|\psi(x)|=1$ for all $x\in G$. We note that it follows from the theorem in [3] that any character can be extended to a semicharacter.

THEOREM. *Let G be a commutative semigroup and let $S\subseteq G$ be a subsemigroup. A character ψ on S can be extended to a character on G if and only if ψ satisfies*:

(*) $\qquad a, b\in S,\ x\in G,$ *and* $ax = bx$ *imply* $\psi(a)=\psi(b)$.

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