THE EXISTENCE OF COMPLETE RIEMANNIAN METRICS

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The purpose of the present note is to prove the following results. *Let M be a connected differentiable manifold which satisfies the second axiom of countability. Then* (i) *M admits a complete Riemannian metric*; (ii) *If every Riemannian metric on M is complete, M must be compact.*

In fact, somewhat stronger results will be given as Theorems 1 and 2 below.

Let M be a connected differentiable manifold. It is known that if M satisfies the second axiom of countability, then M admits a Riemannian metric. Conversely, it can be shown that the existence of a Riemannian metric on M implies that M satisfies the countability axiom. For any Riemannian metric g on M, we can define a natural metric d on M by setting the distance $d(x, y)$ between two points x and y to be the infinimum of the lengths of all piecewise differentiable curves joining x and y. The Riemannian metric g is complete if the metric space M with d is complete. It is known that this is the case if and only if every bounded subset of M (with respect to d) is relatively compact.

We shall say that a Riemannian metric g is *bounded* if M is bounded with respect to the metric d. We shall prove

THEOREM 1. *For any Riemannian metric g on M, there exists a complete Riemannian metric which is conformal to g.*

THEOREM 2. *For any Riemannian metric g on M, there exists a bounded Riemannian metric which is conformal to g.*

The result (ii) mentioned in the beginning is a consequence of Theorem 2, because if a bounded Riemannian metric, which exists on M, is complete, then M itself is compact.

PROOF OF THEOREM 1. At each point x of M, we define $r(x)$ to be the supremum of positive numbers r such that the neighborhood $S(x, r)=\{y; d(x, y)<r\}$ is relatively compact. If $r(x)=\infty$ at some point x, M is compact and hence g is complete. Assume therefore that $r(x)<\infty$ for every x. It is easy to verify that $|r(x)-r(y)| \leqq d(x, y)$ for all x and y in M, which shows that $r(x)$ is a continuous function on

Received by the editors November 14, 1960.

[1] Partially supported by NSF grant 14032.

M. Since M satisfies the second axiom of countability,[2] we can choose a differentiable function $\omega(x)$ such that $\omega(x) > 1/r(x)$ at every point x. We define a conformal Riemannian metric g' by $g'_x = (\omega(x))^2 g_x$ at every point x.

In order to prove that g' is complete, we shall show that $S'(x, 1/3) = \{y; d'(x, y) < 1/3\}$ is contained in $S(x, r(x)/2)$ (and hence relatively compact) for every x, where d' is the distance defined by g'. For this purpose, assume $d(x, y) \geqq r(x)/2$. For any piecewise differentiable curve $x(t)$, $a \leqq t \leqq b$, joining x and y, its length $L = \int_a^b \|dx/dt\| dt$ ($\|dx/dt\|$ denotes the length of the tangent vector dx/dt with respect to g) is not smaller than $d(x, y)$ and hence $L \geqq r(x)/2$. We evaluate the length L' of the same curve with respect to g'. By a mean value theorem, we have

$$L' = \int_a^b \omega(x) \|dx/dt\| dt\, \omega(x(c)) L$$
$$> L/r(x(c)),$$

where c is a number between a and b. Since $|r(x(c)) - r(x)| < d(x, x(c)) \leqq L$, we have $r(x(c)) < r(x) + L$ so that $L' > L/(r(x) + L)$. Since $L \geqq r(x)/2$, we have $L' > 1/3$. Therefore $d'(x, y) \geqq 1/3$. This proves that $S'(x, 1/3)$ is contained in $S(x, r(x)/2)$.

PROOF OF THEOREM 2. By virtue of Theorem 1, we may assume that the given Riemannian metric g is complete. Let o be an arbitrarily fixed point of M. The function $d(x, o)$ is continuous. Let $\omega(x)$ be a differentiable function such that $\omega(x) > d(x, o)$ on M. We shall prove that the Riemannian metric $g' = e^{-2\omega(x)} g$ is bounded. Let x be an arbitrary point of M. Since g is complete, there exists a minimizing geodesic C from o to x, that is, a geodesic C whose length L is equal to $d(x, o)$. Let $x(s)$ be a parametric representation of C in terms of the arc length measured from o. Since any subarc of C is a minimizing geodesic between its end points, we have $d(x(s), o) = s$ for every s. The length of the tangent vector dx/ds with respect to g' is equal to

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[2] This fact, mentioned in the introduction, can be proved, for example, as follows. When M is not compact, we define for every natural number n a neighborhood $U_n(x) = \{y; d(x, y) < r(x)/n\}$ for each point x. It is easy to verify that $\{U_n\}$ defines a uniform structure on the space M and that M is uniformly locally compact (i.e., there is some n, indeed $n = 2$ will do in this case, such that $U_n(x)$ is relatively compact for every x). Since M is connected, it follows that M is the sum of countably many compact subsets. Now for a differentiable manifold, this means that it satisfies the second axiom of countability.

$e^{-\omega(x(s))}$. The length L' of C with respect to g' is thus $\int_0^L e^{-\omega(x(s))}ds$. Since $\omega(x(s)) > d(x(s), o) = s$, we have

$$L' < \int_0^L e^{-s}ds < \int_0^\infty e^{-s}ds = 1,$$

which implies that $d'(x, o) < 1$ for every x.

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