\[ \sum_{j=0}^{m} \sum_{k=0}^{n} \binom{m}{j} \binom{n}{k} \frac{B_j B_k}{m + n - j - k + 1} = (-1)^{m-1} \frac{m! n!}{(m + n)!} B_{m+n} \quad (m + n \text{ even, } mn > 0). \]

**References**


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**MINKOWSKI'S THEOREM ON NONHOMOGENEOUS APPROXIMATION**

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For \( \theta \) irrational, let \( \gamma = \theta a + b \) has no solution in integers \( a \) and \( b \). We give a short proof of Minkowski's classic result that there are infinitely many pairs of integers satisfying \( F < 1/4 \), where \( F = F(\theta, \gamma, x, \gamma) = |x| \cdot |\theta x + y + \gamma| \). First we prove that given any real numbers \( \alpha \) and \( \beta \) there exists an integer \( u \) such that \( |u - \beta| < 1 \) and such that at least one of the following holds:

(A) \( |u - \alpha| \cdot |u - \beta| \leq 1/4 \); \( |u - \alpha| \cdot |u - \beta| \leq |\beta - \alpha| / 2 \).

If \( \beta \) is an integer, set \( u = \beta \). Otherwise define the integer \( n \) by \( n < \beta < n + 1 \). If \( n \leq \alpha \leq n + 1 \) then \( |n - \alpha| \cdot |n + 1 - \alpha| \leq 1/4 \) and similarly for \( \beta \), and so

\[ |n - \alpha| \cdot |n - \beta| \cdot |n + 1 - \alpha| \cdot |n + 1 - \beta| \leq 1/16. \]

Hence \( u = n \) or \( u = n + 1 \) gives inequality (A1). The cases \( n > \alpha \) and \( \alpha > n + 1 \) are symmetric, and we treat \( n > \alpha \). We note that

\[ 2(n - \alpha)^{1/2}(n + 1 - \beta)^{1/2}(\beta - n)^{1/2}(n + 1 - \alpha)^{1/2} \leq (n - \alpha)(n + 1 - \beta) + (\beta - n)(n + 1 - \alpha) = \beta - a \]

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and so (A₂) must hold for \( u = n \) or \( u = n + 1 \).

Now by the pigeon-hole method \([1, p. 1 \text{ or } 2, p. 42]\) it is known that there exist infinitely many pairs of integers \( h, k \) such that 
\[
|k| \cdot |k\theta - h| < 1.
\]
For each such pair choose integers \( r, s \) such that 
\[
|rh - sk + \gamma k| \leq 1/2.
\]
Apply (A) with 
\[
\alpha = \frac{r}{k}
\]
and 
\[
\beta = \frac{(r\theta - s + \gamma)/(k\theta - h)}{\theta},
\]
and define 
\[
x = r - uk, \quad y = -s + uh.
\]
Then we get 
\[
|\theta x + y + \gamma| < |k\theta - h|
\]
and \( F < 1/4 \) from (A₁), \( F \leq 1/4 \) from (A₂). Since \( k\theta - h \) can be made arbitrarily small, and since \( \theta x + y + \gamma \neq 0 \), we get infinitely many pairs \( x, y \) satisfying \( F \leq 1/4 \). But at most one pair can give \( F = 1/4 \), because 
\[
\theta x_1 + y_1 + \gamma = \pm (4x_1)^{-1} \quad \text{and} \quad \theta x_2 + y_2 + \gamma = \pm (4x_2)^{-1}
\]
would imply the rationality of \( \theta(x_1 - x_2) + y_1 - y_2 \) and hence of \( \theta \). This proof can be readily extended to Minkowski’s theorem on the product of two linear forms, as will be shown elsewhere. A proof that \( 1/4 \) is the best possible constant is given in \([1, p. 49]\).

References


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