STARLIKE HYPERGEOMETRIC FUNCTIONS

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Let $a$, $b$, $c$ be complex numbers with $c \neq 0$, $-1$, $-2$, \ldots. For the hypergeometric function $F(a, b, c; z)$, which is defined in $|z| < 1$ by the series

$$1 + \frac{ab}{c!} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 + \cdots,$$

the identities

$$cF(a, b+1, c; z) = aF(a+1, b+1, c+1; z)$$

$$+ (c-a)F(a, b+1, c+1; z),$$

$$cF'(a, b, c; z) = abF(a+1, b+1, c+1; z)$$

are readily established.

When in the identity

$$c(1-z)F(a, b+1, c; z) = cF(a, b, c; z) - (c-a)zF(a, b+1, c+1; z),$$

which is due to Gauss [1, p. 130], $cF(a, b+1, c; z)$ is replaced by the right member of (2) the identity

$$a(1-z)F(a+1, b+1, c+1; z)$$

$$= cF(a, b, c; z) - (c-a)F(a, b+1, c+1; z)$$

is obtained. By (3) this can be rewritten for $z \neq 1$ as

$$z \left[ \frac{F'(a, b, c; z)}{F(a, b, c; z)} \right] = \frac{bz}{1 - z} \left[ 1 - \frac{c-a}{c} \frac{F(a, b+1, c+1; z)}{F(a, b, c; z)} \right],$$

which is instrumental in the proof of the following:

**Theorem.** Let $0 < a < c$ and $-1 < b \leq a$. Then the function $u(z) = zF(a, b, c; z)$ is univalent and starlike with respect to the origin for $|z| < \rho$, where

(i) $$\rho = \frac{1}{1-b} \quad \text{when} \quad -1 < b < 0;$$
(ii) \( \rho = 1 \) when \( 0 \leq b \leq 2 \);

(iii) \( \rho = \frac{1}{b-1} \) when \( b > 2 \).

**Proof.** A sufficient condition for \( u(z) \) to be univalent and starlike in \( |z| < r \) is \([4]\)

\[
(5) \quad \Re s(z) \geq 0, \quad |z| < r,
\]

where

\[
(6) \quad s(z) = \frac{u'(z)}{u(z)} = \frac{F'(a, b, c; z)}{F(a, b, c; z)} + 1.
\]

A result of Paydon and Wall \([2]\) (see also \([5, ~p.~46]\)) shows that if \( 0 < g_n < 1 \) \( (n = 1, 2, \ldots) \), then the function \( f(z) \) defined in \( |z| < 1 \) by the continued fraction

\[
(7) \quad f(z) = \frac{g_1}{1 - \frac{(1 - g_1)g_2z}{1 - \frac{1}{1 - \frac{1}{1 - \frac{1}{\ldots}}}}}
\]

satisfies

\[
(8) \quad \left| \frac{f(z) - 1}{2 - g_1} \right| \leq \frac{1 - g_1}{2 - g_1}, \quad |z| < 1.
\]

A result of Wall \([5, p.~283]\) states that when \( f(z) \) is given by \(7)\),

\[
(9) \quad \frac{1 - f(z)}{1 - zf(z)} = \frac{1 - g_1}{1 - \frac{1}{1 - \frac{1}{1 - \frac{1}{\ldots}}}}
\]

where the right member of \(9)\) is obtained from the right member of \(7)\) by replacing \( g_n \) by \( 1 - g_n \). In this case inequality \(8)\) becomes

\[
\left| \frac{1 - f(z)}{1 - zf(z)} - \frac{1}{1 + g_1} \right| \leq \frac{g_1}{1 + g_1}, \quad |z| < 1,
\]

which is equivalent to the inequality

\[
(10) \quad \left| \frac{f(z) - 1}{g_1(1 - z) + g_1 - z - \bar{z} + (1 - g_1)z\bar{z}} \right| \leq \frac{g_1 |1 - z|}{1 + g_1 - z - \bar{z} + (1 - g_1)z\bar{z}}, \quad |z| < 1.
\]
The function \((c - a) F(a, b + 1, c + 1; z)/c F(a, b, c; z)\) is identified with the function \(f(z)\) of (7) \([5, p. 339]\) and

\[
s(z) = 1 + \frac{b z}{1 - z} [1 - f(z)].
\]

The requirement \(0 < g_n < 1\) is equivalent to the conditions \(0 < a < c, -1 < b < c\), and the second condition can be replaced by \(-1 < b \leq a\) since \(F(a, b, c; z)\) is symmetric in \(a\) and \(b\). Thus, under the hypotheses of the Theorem, it follows from (10) that

\[
|s(z) - C| \leq R, \quad |z| < 1,
\]

where

\[
C = 1 + \frac{b z (c - a \overline{z})}{2c - a + c(z + \overline{z}) + a z \overline{z}},
\]

\[
R = \frac{|b z| (c - a)}{2c - a + c(z + \overline{z}) + a z \overline{z}},
\]

and hence (5) will hold if the function

\[
H(\theta) = \Re C - R
\]

is non-negative for \(z = re^{i\theta}, r < 1\).

When \(-1 < b < 0,\)

\[
H(\theta) = 1 + \frac{b r (c \cos \theta - ar + c - a)}{2c - a - 2cr \cos \theta + ar^2},
\]

and a simple computation shows that the minimum of \(H(\theta)\) occurs for \(\theta = 0\). The statement (i) is then easily obtained from the requirement

\[
H(0) = 1 + \frac{br}{1 - r} \geq 0.
\]

Similarly, when \(b > 0,\) the minimum of

\[
H(\theta) = 1 + \frac{b r (c \cos \theta - ar - c + a)}{2c - a - 2cr \cos \theta + ar^2}
\]

occurs when \(\theta = \pi\) and the requirement

\[
H(0) = 1 - \frac{br}{1 + r} \geq 0
\]
leads to the statements (ii) and (iii).

It is evident that the result (ii) is sharp since \(u(z)\) is not regular at \(z = 1\) except in the case where \(b = 0\).

A necessary and sufficient condition for a nonconstant regular function \(F(z)\) to be univalent in \(|z| < \rho\) and to map \(|z| < \rho\) onto a convex region is that \(zF'(z)\) be univalent and starlike with respect to the origin for \(|z| < \rho\) \([3, p. 221]\). Since

\[czF'(a, b, c; z) = abzF(a + 1, b + 1, c + 1; z)\]

by (3), application of the Theorem yields the following:

**Corollary.** The function \(F(a, b, c; z)\), where \(-1 < a < c, -2 < b \leq a, a \neq 0, b \neq 0, c \neq 0\), is univalent in \(|z| < \rho\) and maps \(|z| < \rho\) onto a convex region if

(i) \(\rho = -1/b\) when \(-2 < b < -1,\)

(ii) \(\rho = 1\) when \(-1 \leq b \leq 1, b \neq 0,\)

(iii) \(\rho = 1/b\) when \(b > 1.\)

**References**


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