CLASSES OF $p$-VALENT STARLIKE FUNCTIONS

G. R. BLAKLEY

1. Introduction. The winding number associated with a starlike function exhibits a certain monotonicity property (Theorem 1 below). This property is used to show that several alternatives to the definition of the class $S(p)$ of $p$-valent starlike functions are trivial. From it there also follows a simple and explicit example of a coefficient problem in $S(p)$ with no solution. This situation, which Goodman has treated in some detail [2], is interesting since such problems always have solutions in the schlicht case [3; 4].

Let $S$ be the class of all functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ which are regular and schlicht in $|z| < 1$ and let $S^*$ be the subclass of $S$ consisting of those functions whose image domains are starshaped with respect to the origin. For a given positive integer $p$ let $S(p)$, the class of $p$-valent starlike functions, be the class of all functions $f$ to which there corresponds some $r$, $0 < r < 1$, such that for any $z$, $r < |z| < 1$, $\Re \{zf''(z)/f'(z)\} \geq 0$ and $(1/2r) \int_a^b \Re \{zf''(z)/f(z)\} \, dt = p$, $z = qei\theta$, for each $q$, $r < q < 1$. This integral is just the number of zeros of $f$ in the interior of the circle $|z| = q$ and hence $f$ has $p$ zeros in the open unit disk, and is in fact $p$-valent there [2]. In a certain sense the classes $S(1)$ and $S^*$ coincide, i.e., if $f \in S^*$ then $f \in S(1)$ and if $f \in S(1)$ then $f/f'(0) \in S^*$.

2. The winding number. If $Q$ is a path in $U = \{z: |z| < 1\}$ and $f$ is analytic in $U$, let $f(Q)$ denote the path which is the image of $Q$ under $f$ and which has the induced orientation. The properties of the winding number

$$n[f(Q), a] = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z) - a} \, dz$$

are well known. In this paper $f$ will lack singularities and $Q$ will be a circle. Hence $n[f(Q), a]$ will be the number of times $f$ assumes the value $a$ in the open disk bounded by $Q$.

If a function $f$ is regular at a point $a \neq 0$ and $\Re \{zf''(z)/f'(z)\} \geq 0$ for all $z$ in some neighborhood $M$ of $a$ then the fact that $\Re \{zf''(z)/f(z)\}$ is harmonic in some neighborhood $N \subset M$ of $a$ implies that $\Re \{af''(a)/f(a)\} > 0$ and, consequently, also that $f'(a) \neq 0$. From this fact, which will be of frequent use below, follows

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152
Theorem 1. Let the function $f$ be regular in the open unit disk $U$ and zero at the origin. Suppose there is an $r = r(f)$, $0 < r < 1$, such that (i) $f(z) \neq 0$ and (ii) $\Re \{z f''(z)/f'(z)\} \geq 0$, whenever $r < |z| < 1$. Then for any $q$, $r < q < 1$, and associated $Q = \{z: |z| = q\}$ and any $a \in U$, the winding number $n[f(Q), sf(a)]$ is a decreasing function of the positive real variable $s$ as long as $sf(a) \in f(U)$.

Proof. Let $f(U)$ be the image of $U$ under $f$. The winding number $n[f(Q), sf(a)]$ is a nonnegative integer (hence real) and is constant throughout each component of $f(U)$ determined by $f(q)$. Therefore, as has just been noted, $\Re \{z f''(z)/f'(z)\} = \partial \arg f(z)/\partial \arg z > 0$ whenever $r < |z| < 1$, i.e. $\arg f$ is a strictly increasing function of $\arg z$ for $z \in Q$. Furthermore the fact that $f$ is never zero in $\{z: r < |z| < 1\}$ and has only a finite number $m > 0$ of zeros in $\{z: |z| \leq r\}$ implies, with the help of the argument principle, that $\arg f(z)$ increases by $2\pi m$ as $z$ makes one positively directed circuit of $Q$. Thus $\arg f(z)$ takes on each value $b$, $0 \leq b < 2\pi$ precisely $m$ times as $z$ traverses $Q$.

If $a \in Q$ is arbitrary it is apparent that the angle $\theta$ from the radius vector $f(a) - 0$ to the vector tangent to $f(Q)$ at $f(a)$ lies in the interval $0 < \theta < \pi$. The geometric meaning of the winding number now makes it obvious that its value falls as $f(Q)$ is crossed in an outward direction and in fact that this decrease is just some integer $n$, $1 \leq n \leq m$, which is the number of points of $Q$ mapped into $f(a)$ by $f$. The proof of the theorem is now complete.

3. The class $M^*$.

Definition 1. Let $p$ be fixed, $\geq 1$. Suppose the function $h(z) = z^p + b_{p+1}z^{p+1} + b_{p+2}z^{p+2} + \cdots$ is regular in $U$ and satisfies the conditions

(i) $h$ is $p$-valent in $U$, and

(ii) there exists $r = r(h)$, $0 < r < 1$, such that $\Re \{zh'(z)/h(z)\} \geq 0$ whenever $r < |z| < 1$.

Then $h$ is called a function of class $M^*_p$.

Manifestly if $h \in M^*_p$ then for any $q$, $r < q < 1$, and $Q = \{z: |z| = q\}$ it must be that $n[h(Q), 0] = p$, i.e., $(1/2\pi)\int_{0}^{2\pi} \Re \{zh'(z)/h(z)\} dt = p$, $z = ge^{it}$, so that $h \in S(p)$. Thus $M^*_p \subset S(p)$. Let $(S^*)^p$ be the class of $p$th powers of functions of $S^*$. Then

Theorem 2. $M^*_p = (S^*)^p$, $p = 1, 2, 3, \ldots$.

Proof. Choose any function

$$h(z) = z^p + b_{p+1}z^{p+1} + b_{p+2}z^{p+2} + \cdots$$

$$= z^p(1 + b_{p+1}z + b_{p+2}z^2 + \cdots) = z^p g(z)$$
of \( M_p^* \). Note that \( g \) is regular and nonzero in \( U \). Hence so is \( \left[ g(z) \right]^{1/p} \). Since \( g(0) = 1 \) the function \( \left[ g(z) \right]^{1/p} \) can be assumed to be 1 at the origin. Set \( f(z) = zg(z)^{1/p} = z + \cdots \). Then \( \left[ f(z) \right]^{p} = h(z) \). Furthermore, if \( z \in U \), \( zh'(z)/h(z) = p \left( f(z)/f(z) \right) \) whence, since \( p > 0 \), \( \Re \left[ zf'(z)/f(z) \right] \geq 0 \) whenever \( r(h) = r < |z| < 1 \). Hence, just as above, \( \arg f(z) \) is a strictly increasing function of \( \arg z \) for \( z \in \Omega \). Therefore \( f(\Omega) \) is a simple closed curve and by Darboux’s Theorem \( f \) is schlicht. Thus \( f \in S^* \), and consequently \( M_p^* \subset (S^*)^p \).

If, now, \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \) is a member of \( S^* \) and the function \( h(z) = z^p + \cdots \) is defined by setting \( h(z) = \left[ f(z) \right]^p \) then certainly \( h \in M_p^* \), for \( f \) can take on each \( p \)th root of a given number at most once in \( U \), showing that \( h \) satisfies condition (i). That \( h \) satisfies condition (ii) has already been shown in the first half of this proof. Therefore \( (S^*)^p \subset M_p^* \) and the theorem is proved.

4. The class \( N_p^* \).

Definition 2. Let \( p \) be fixed, \( p \geq 1 \). Let \( m \) be an integer, \( 1 \leq m \leq p \). Suppose the function \( h(z) = z^m + b_{m+1}z^{m+1} + b_{m+2}z^{m+2} + \cdots \) is regular in \( U \) and satisfies the conditions

(i) \( h \) is at most \( p \)-valent in \( U \), and

(ii) \( \Re \left[ zh'(z)/h(z) \right] \geq 0 \) for all \( z \in U \).

Then \( h \) is called a function of class \( N_p^* \).

The relationship of the class \( N_p^* \) to \( S(p) \) is made plain by the following

**Theorem 3.** \( N_p^* = S^* \cup (S^*)^2 \cup \cdots \cup (S^*)^p \), \( p = 1, 2, 3, \ldots \)

**Proof.** Consider an arbitrary \( h \in N_p^* \). The function \( h \) is of the form \( h(z) = z^m + b_{m+1}z^{m+1} + b_{m+2}z^{m+2} + \cdots \) for some integer \( m \), \( 1 \leq m \leq p \). To show that \( h \in M_p^* = (S^*)^m \) it clearly suffices to verify that \( h \) is \( m \)-valent in \( U \). Consider any point \( a \neq 0 \) of \( U \). If \( h(a) = 0 \) then, in some neighborhood of \( a \), \( h(z) = (z-a)^n g(z) \) where \( 1 \leq n \) and \( g(a) \neq 0 \) (for the identically zero function is not a member of \( N_p^* \)). Then \( zh'(z)/h(z) = na/(z-a) + n + zg'(z)/g(z) \). Certainly \( n + zg'(z)/g(z) \) is analytic at \( z = a \) since \( g(a) \neq 0 \) and therefore \( zh'(z)/h(z) \) has a pole of order 1 at \( z = a \) in contradiction to condition (ii) in the hypothesis concerning \( h \). Thus \( h \) can be zero in \( U \) only at the origin. Hence for any \( q \), \( 0 < q < 1 \), and associated \( Q = \{ z : |z| = q \} \) and \( G = \{ z : |z| < q \} \), it is apparent that \( n \left[ h(Q), h(0) \right] = n \left[ h(Q), 0 \right] = m \). Theorem 1 now guarantees that \( n \left[ h(Q), h(z) \right] \leq m \) whenever \( z \in G \). Since \( q \) can be arbitrarily close to 1, \( h \) must be \( m \)-valent in \( U \). Thus \( N_p^* \subset S^* \cup (S^*)^2 \cup \cdots \cup (S^*)^p \).

For \( m = 1, 2, 3, \ldots, p \) the proof that \( (S^*)^m \subset N_p^* \) is the same as
the corresponding part of Theorem 2. This establishes the opposite inclusion and therewith the theorem.

5. The class $S^*_p$.

**Definition 3.** Let $p = 1, 2, 3, \ldots$. Suppose the function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is regular in $U$ and satisfies the conditions

(i) $f$ is at most $p$-valent in $U$,

(ii) there exists some $a \in f(U)$ such that $f(z) = a$ exactly $p$ times in $U$, and

(iii) there exists $r = r(f)$, $0 < r < 1$, such that $\Re\{zf'(z)/f(z)\} \geq 0$ whenever $r < |z| < 1$.

Then $f$ is called a function of class $S^*_p$.

Evidently $S^*_1 = S^*$ which, in turn, is in the sense above alluded to just equal to $S(1)$. But not even in this sense is $S^*_p = S(p)$. However

**Theorem 4.** $S^*_p \subseteq S(p)$, $p = 1, 2, 3, \ldots$.

**Proof.** If $f \in S^*_p$ then all that must be verified to show that $f \in S(p)$ is the existence of a $d$, $0 < d < 1$, such that whenever $d < q < 1$

$$n[f(Q), 0] = \frac{1}{2\pi} \int_0^{2\pi} \Re \left\{ \frac{zf'(z)}{f(z)} \right\} \, dt = p,$$

$$z = qe^{it}.$$ 

There are some $a \in U$, some number $k$, $0 < k < 1$, and a circle $K = \{z: |z| = k\}$ contained in $U$ such that $n[f(K), f(a)] = p$. Let $c$, $0 < c < 1$, be the largest of the moduli of the (at most $p$) points $z_i$ of $U$ at which $f(z_i) = 0$. The number $r = r(f)$ is already associated with $f$. Define $d = \max\{r, k, c\}$. Then for any $q$, $d < q < 1$, and associated $Q = \{z: |z| = q\}$ it is true that $n[f(Q), f(a)] = p$. The fact that $f$ is at most $p$-valent in $U$ implies that $n[f(Q), 0] \leq p$. Hence an application of Theorem 1 to $f$ yields $\rho = n[f(Q), f(a)] \leq n[f(Q), 0] \leq p$. Since $q$ can be arbitrarily close to 1 this shows that $f \in S(p)$, i.e. $S^*_p \subseteq S(p)$, which completes the proof.

If $g$ is any function of $S(p)$ having a simple zero at the origin then $g$ can be written $g(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$. It is now a trivial matter to verify that

$$\frac{g(z)}{a_1} = \frac{g(z)}{g'(0)} = z + \frac{a_2}{a_1} z^2 + \frac{a_3}{a_1} z^3 + \cdots$$

is an element of $S^*_p$. And in this sense, the same as with the classes $S(1)$ and $S^* = S^*_1$ (i.e. except for normalization) the class $S^*_p$ is just the class of all functions of $S(p)$ having a simple zero at the origin.
It is clear that for \( p = 2, 3, 4, \ldots \) no function of \( S_p^* \) is the power of a (schlicht) starlike function.

The following theorems give an example of a coefficient problem which has no solution.

**Theorem 5.** Let \( p \) be fixed, \( p \geq 2 \). If \( k \) is any complex number such that \( 2 - 1/p < |k| \) then the function \( f(z) = z + k z^p \) is a member of \( S_p^* \).

**Proof.** The function \( f \) has \( p \) zeros in \( U \) and is obviously \( p \)-valent in \( U \). Also, for \( |z| = 1 \),

\[
\Re\left\{ \frac{zf'(z)}{f(z)} \right\} = \Re\left\{ \frac{p + (1 - p)z}{z + k z^p} \right\} \geq p - \frac{p - 1}{|k| - 1} \]

\[
= \frac{p}{k - 1} \left( |k| - 2 + \frac{1}{p} \right) > 0,
\]

which persists, by continuity, for \( r < |z| < 1 \), for some \( r, 0 < r < 1 \). Therefore \( f \in S_p^* \).

**Theorem 6.** Let \( p \) be fixed, \( p \geq 2 \). Let \( n \neq p \) be chosen, \( n \geq 2 \). Then to any complex number \( q \) there corresponds a function \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \) of \( S_p^* \) for which \( a_n = q \).

**Proof.** If \( q = 0 \) the theorem is an immediate consequence of Theorem 5. So consider arbitrary fixed nonzero \( g \), and let \( b \) be any complex number whose modulus satisfies the inequality \( 1 + |q| < p + |q| + |n - p| \cdot |q| < |b| \). Now consider the function \( f(z) = z + b z^p + q z^n \). In consequence of the above inequality and a theorem of Pellet [1, p. 10] on roots of polynomials it follows immediately that \( n[f(C), 0] = p \), where \( C = \{z: |z| = 1\} \). The fact that \( \max_{z \in C} |zf'(z)/f(z) - p| < 1 < p \) implies, just as in the proof of Theorem 5, that there exists \( d, 0 < d < 1 \), such that \( \Re\{zf'(z)/f(z)\} \geq 0 \) whenever \( d < |z| < 1 \). Since \( f \) takes on only \( p \) zeros in \( U \) there exists \( t, 0 < t < 1 \), such that \( f(z) \neq 0 \) whenever \( t < |z| < 1 \). Set \( r = \max(d, t) \). Then \( 0 < r < 1 \) and Theorem 1 is applicable to \( f \). Hence if \( r < x < 1 \) and \( X = \{z: |z| = x\} \) the winding number \( n[f(X), sf(a)] \) is a decreasing function of the positive real variable \( s \) whenever \( sf(a) \in f(U) \). Thus \( p = n[f(C), 0] = n[f(X), 0] \geq n[f(X), k] \) for any \( k \in f(U) \). But \( x \) can be arbitrarily close to \( 1 \). Therefore \( f \in S_p^* \) and the theorem is proved.

Consideration of the classes of \( p \)-valent starlike functions treated above has given rise to the following question concerning a decomposition for elements of \( S(p) \). Given any \( f \in S(p) \), does \( f \) have a representation \( f = gh \) where \( g \in S_p^* \), \( h \in (S^*)^{p-1} \)?

