THE POWER OF TOPOLOGICAL TYPES OF SOME
CLASSES OF 0-DIMENSIONAL SETS

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By a result of Mazurkiewicz and Sierpinski, there exist \( \aleph_1 \) topological types of compact and countable sets. Since a countable set is 0-dimensional, there arises a natural question: what is the power of topological types of other classes of 0-dimensional sets? In this paper we consider separable metric spaces only. Every 0-dimensional space being topologically contained in the Cantor set \( C \), we confine ourselves to subsets of this set.

We prove the following three theorems:

**Theorem 1.** There exist two topological types of open subsets of the Cantor set \( C \).

**Theorem 2.** There exist \( \aleph_0 \) topological types of closed subsets of the Cantor set \( C \).

**Theorem 3.** There exist \( \ell \aleph_0 \) topological types of 0-dimensional \( G_\delta \) sets which are dense in themselves.

Theorem 1 is known in part, but it seems to the author that an exact proof of it has not been published so far. Theorems 2 and 3 are new; the latter gives an answer to a problem by Knaster and Urbanik.

The paper contains also some lemmas on homeomorphisms and a notion of a rank \( r_p(B) \) of a point \( p \) relative to the set \( B \).

1. In this section a lemma on homeomorphisms and the above Theorem 1 are proved.

**Lemma 1.** Let \( \{ F_n \} \) and \( \{ G_n \} \) be two sequences of sets satisfying

1. \( F_n \cap F_m = 0 = G_n \cap G_m \) for \( n \neq m \),
2. for every \( n \) the set \( F_n \) is open in the union \( F = \bigcup_{n=1}^\infty F_n \) and \( G_n \) is open in \( G = \bigcup_{n=1}^\infty G_n \), and
3. for every \( n \) there exists a homeomorphism \( h_n \) such that \( h_n(F_n) = G_n \), \( n = 1, 2, \ldots \).

Then the mapping \( h \) defined by \( h(x) = h_n(x) \) for \( x \in F_n \) is a homeomorphism between \( F \) and \( G \).

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1 See [6, p. 22].
2 See [4, p. 173].
3 Some general hints may be found in [3, p. 198].
4 See [3, p. 198].
Proof. By (1) and (3), $h$ is a one-to-one mapping of $F$ onto $G$. Since the proofs of the continuity of $h$ and $h^{-1}$ are symmetric, we shall show that $h$ is continuous.

Indeed, let $\{x_n\}$ be a sequence of points belonging to $F$, tending to a point $x$ of $F$: $x_n \to x$. Since $x \in F$, there exists a number $n_0$ such that $x_n \in F_{n_0}$. Now by (2) there exists a number $N$ such that for $n > N$ there is $x_n \in F_{n_0}$ (since otherwise the set $F_{n_0}$ would not be open in $F$). But $h_{n_0}$ is continuous—as a homeomorphism—and therefore for $n > N$:

$$h(x_n) = h_{n_0}(x_n) \to h_{n_0}(x) = h(x).$$

Remark 1. Let $F_n$ be the plane set defined by $F_n = \{(x, y) : x = 1/n, 0 < y < 1\}$ and put $G_n = \{(x, y) : x = 0; 0 < y < 1\}$ and $G_{n+1} = F_n$, $n = 1, 2, \ldots$. For these sets the assumption (2) of the lemma is not satisfied for the set $G_1$ only and evidently $F = \bigcup_{n=1}^{\infty} F_n$ is not homeomorphic with $G = \bigcup_{n=1}^{\infty} G_n$, since $G$ is a compact set and $F$ is not. This shows also that assumption (2) of the lemma cannot be replaced by the assumption that $F_n$ and $G_n$ are compact and $\rho(F_n, F_m)$ and $\rho(G_n, G_m)$ are positive for all $n \neq m$.

To prove Theorem 1 it suffices to show that:

Every open subset of the Cantor set $C$ is either homeomorphic to $C$ or to $C$ without the zero point: $C \setminus \{0\}$.

Proof. Let $G$ be an open subset of the Cantor set $C$. Then $G$ can be written in the form:

$$G = G_1 \cup G_2 \cup \cdots, G_n \cap G_m = 0 \text{ for } n \neq m,$$

where the sets $G_n$ are closed and open in $C$.

Now two cases are possible:

(a) $G$ is a finite union of the sets $G_n$, i.e. there exists an integer $N$ such that $G_n = 0$ for $n > N$, and

(b) all the sets $G_n$ in (4) are nonempty.

Since

(5) a closed and open subset of the Cantor set $C$ is a perfect set, we see that in case (a) the set $G$ is a perfect 0-dimensional set and therefore homeomorphic to the Cantor set $C$.

In case (b) we can write the set $C \setminus \{0\}$ analogically as in (4) in the form:

$$C \setminus \{0\} = F_1 \cup F_2 \cup \cdots, F_n \cap F_m = 0 \text{ for } n \neq m,$$

where the sets $F_n$ are nonempty and closed and open in $C$.

By (5) there exists for every $n$ a homeomorphism $h_n$ between $F_n$ and $G_n$ and therefore by (4) and (6) the assumptions of the lemma hold.

* By $\rho(F_n, F_m)$ we understand the distance between the sets $F_n$ and $F_m$, i.e. $\rho(F_n, F_m) = \inf_{x \in F_n, y \in F_m} \rho(x, y)$, where $\rho(x, y)$ denotes the distance between the points $x$ and $y$.

* See [4, p. 166].
Thus by the lemma the set \( G \) is, in case (b), homeomorphic to \( C \setminus \{0\} \).

**Remark 2.** Theorem 1 may also be proved in another way by using the one-point compactification theorem,\(^7\) but such an exact proof is not simpler than ours.

2. We show in this section that there exist \( 2^{N_0} \) topological types of closed subsets of the Cantor set \( C \). Since the power of all closed subsets of \( C \) is \( 2^{N_0} \) and every 0-dimensional space has a topological image in the Cantor set \( C \), it suffices to construct a family \( \mathcal{F} \) of power \( 2^{N_0} \) of compact, 0-dimensional sets, such that no two sets belonging to this family are homeomorphic. To do this we introduce the notion of a rank \( r_p(B) \) of a point \( p \) relative to the set \( B \). First we recall the notion of the coherence and adherence of a set \( E \) in the sense of Hausdorff.\(^8\)

The 0th coherence of \( E \) is equal to \( E \); the \( \alpha \)th coherence of \( E \) is the set of all limits \( x = \lim_{n \to \infty} x_n \); \( x_i \neq x_j \) for \( i \neq j \) such that \( x \) and \( x_n \) belong to the \( (\alpha - 1) \)th coherence, if \( \alpha - 1 \) exists, and the intersection of all coherences with indices \( <\alpha \) if \( \alpha \) is a limit number. The \( \alpha \)th adherence is the difference between the \( \alpha \)th and the \( (\alpha + 1) \)th coherences.

Evidently, the \( \alpha \)th adherence is an isolated set. The \( \alpha \)th adherence of the set \( E \) will be denoted by \( E_{(\alpha)} \). It is clear that if \( E \) is a compact and countable set and \( E^{(\beta)} \) is the last derivative\(^9\) \((\neq 0)\) of \( E \), then \( E^{(\beta)} = E_{(\beta)} \) and \( E = \bigcup_{1 \leq \beta} E_{(\beta)} \).

**Example 1.** Take on the \( x \)-axis the sets of points defined by: \( E_1 = \{ x; x = 1/n, n = 1, 2, \ldots \} \), \( E_2 = \{ x; x = 1/n + 1/m, m, n = 1, 2, \ldots \} \), \( E_3 = E_2 \setminus E_1 \cup \{ 0 \} \). Then, the first coherence of \( E_1 \) is empty and the first derivative of \( E_1 \) consists of the point \( x = 0 \). The first coherence of the set \( E_2 \) consists of the point 0. The second coherence of \( E_3 \) is empty. The first derivative of \( E_2 \) is the set \( E_1 \cup \{ 0 \} \) and the second derivative of \( E_3 \) consists of the point 0.

We define now the rank \( r_p(B) \) of a point \( p \in \overline{B}^{10} \) where \( B \) is a countable set such that \( \overline{B} \) is 0-dimensional.\(^{11}\)
Definition. Let \( p \in B \) where \( B \) is a countable set and \( B \) is 0-dimensional. If \( p \in B \) \((0)\) we define \( r_p(B) = 0 \). If there exists an \( \alpha \) such that \( p = \lim_{n \to \infty} p_n \) where \( p_n \in B \langle \alpha \rangle \) and \( p \) is not a limit point\(^{11} \) of \( B \langle \alpha + 1 \rangle \), we define \( r_p(B) = \alpha + 1 \).

If such an \( \alpha \) does not exist, then there exist an ordinal \( \alpha' \), a sequence \( \{ \alpha_n \} \) of ordinals such that \( \alpha_n \to \alpha' \) and a sequence of points \( p_n \in B \langle \alpha_n \rangle \) such that \( p = \lim_{n \to \infty} p_n \) and \( p \) is not a limit point of \( B \langle \alpha' \rangle \). In this case we define \( r_p(B) = \alpha' \).

Example 2. If \( E_3 \) is the set defined in Example 1, the rank of the point 0 relative to \( E_3 \) is equal to 1.

Let now \( E_1 \) and \( E_2 \) be compact and countable sets, such that the \( \omega \)th derivative \( E_1^\omega \) of \( E_1 \) consists of the point \( p \): \( E_1^\omega = \{ p \} \) and the second derivative \( E_2^2 \) of \( E_2 \) consists of the point \( q \): \( E_2^2 = \{ q \} \). Put \( E_3 = E_1 \times (q) \cup (p) \times E_2 \) and \( B = [(p) \times E_2] \setminus (p, q) \). Then \( r_{(p, q)}(B) = 2 \) and \( r_{(p, q)}(E_3) = \omega \).

To define the family \( \mathcal{F} \) a few additional simple remarks are needed.

Since the order \( \alpha \) of a coherence is an invariant of homeomorphisms, it is easily seen that

(7) the rank \( r_p(B) \) is an invariant of homeomorphisms defined on \( B \).

Take now the Cantor set \( C \) and let \( E \) be a compact and countable subset of \( C \) such that the \( \omega \)th derivative \( E^\omega \) of \( E \) consists of the point \( q \): \( E^\omega = \{ q \} \). Take the \( n \)th adherence \( E_{(n)} \) of \( E \), \( n = 1, 2, \ldots \) and choose from every \( E_{(n)} \) a point \( p_n \).

Since the order of an adherence is invariant under homeomorphisms we have that

(8) if \( h \) is any homeomorphism of \( E \) into itself, then \( h(p_n) \neq p_m \) for \( n \neq m \).

Let now \( D_n \) be the sequence of intervals in the plane defined by \( D_n = \{ (x, y) \mid x = p_n, 0 \leq y \leq 1 \} \) \( n = 1, 2, \ldots \) and let \( \{ \alpha_n \} \) be a sequence of ordinals: \( 1 < \alpha_n < \omega \). Choose in every \( D_n \) a countable and compact subset \( F_n \) such that \( \alpha_n \) be the order of the last derivative \( F_n^\omega \) of \( F_n \) and that \( F_n^\omega = \{ p_n \} \). Then the set \( A = C \cup \bigcup_{n=1}^\infty F_n \) is compact (since the diameters of \( D_n \) are equal to \( 1/n \) and \( F_n \subset D_n \)) and 0-dimensional. By the definition of \( F_n \) we have also

(9) \( r_{p_n} \left( \bigcup_{n=1}^\infty F_n \setminus E \right) = \alpha_n > 1, \quad n = 1, 2, \ldots \).

Now take in the plane an arbitrary bounded and isolated set \( I \)

\(^{11}\) A point \( x \) such that there exists a sequence \( \{ x_n \} \) of points \( x_n \) belonging to \( E \), \( x_n \neq x_m \) for \( n \neq m \) and such that \( x_n \to x \) is called a limit point of \( E \).

\(^{12}\) \( \times \) denotes the Cartesian product and \( (p, q) \) is the point in the Cartesian product.
disjoint with $C$ such that $I^{(1)} = E$. Then the set $A_1 = C \cup \bigcup_{n=1}^{\infty} F_n \cup I$ is 0-dimensional and compact. Denoting the decomposition of $A_1$ according to the theorem of Cantor-Bendixson by $A_1 = P_1 \cup B_1$ with $P_1$ as perfect set, we obtain

$$P_1 = C \quad \text{and} \quad B_1 = \left( \bigcup_{n=1}^{\infty} F_n \cup I \right) \setminus E$$

and by the definition of $I$,

$$P_1 \cap B_1 = E.$$

Since $I$ is isolated there is also, by (9),

$$r_\beta(B_1) = \alpha_n > 1 \quad \text{and for every} \quad p \in E \quad \text{and} \quad p \neq p_n, \quad r_\beta(B_1) = 1.$$

If we take now any other sequence $\{ \beta_n \}$ of ordinals: $1 < \beta_n < \Omega$ and the same set $E$ and points $p_n$, we can construct, analogically as before, a 0-dimensional and compact set $A_2$ with the following properties:

If we denote the decomposition of $A_2$ according to the theorem of Cantor-Bendixson by $A_2 = P_2 \cup B_2$ with $P_2$ as perfect set, then

$$P_2 = C \quad \text{and} \quad P_2 \cap B_2 = E.$$

Also

$$(10') \quad r_\beta(B_2) = \beta_n > 1 \quad \text{and for every} \quad p \in E \quad \text{and} \quad p \neq p_n, \quad r_\beta(B_2) = 1.$$ 

Now suppose that there exists a homeomorphism $h$ between $A_1$ and $A_2$: $h(A_1) = A_2$. Then we would have $h(P_1 \cap B_1) = P_2 \cap B_2$, i.e., $h(E) = E$. Hence by (8) there would be $h(p_n) \neq p_m$ for $n \neq m$. But, by (7), (10) and (10') there must be $h(p_n) = p_n$ and therefore by (7), $\alpha_n = \beta_n$ for every $n$. This shows that if the sequences $\{ \alpha_n \}$ and $\{ \beta_n \}$ are different, the sets $A_1$ and $A_2$ cannot be homeomorphic. But the power of all sequences $\{ \alpha_n \}$, $1 < \alpha_n < \Omega$ is $\aleph_0 = 2^{\aleph_0}$. Hence Theorem 2 holds.

Remark 3. In [7, p. 119], we introduced a function $\sigma_B(A)$ assigning to every 0-dimensional compact set $A$ an ordinal $< \Omega$. Using this function, it can be easily shown that the power of all topological types of compact uncountable subsets of the Cantor set is $\aleph_1$. (This can be also obtained from the result of Mazurkiewicz and Sierpinski, mentioned at the beginning of this paper.) Thus by the continuum hypothesis it is equal to $2^{\aleph_0}$, but we proved this fact without recourse to this hypothesis.

Note also that the fact that there exist $2^{\aleph_0}$ topological types of closed sets (not necessarily 0-dimensional) was stated in [6, p. 27].

3. In this section a proof of Theorem 3 is given. Two lemmas are also proved.
Lemma 2. Let $C_i$ and $C_2$ be two compact $0$-dimensional sets and let $S_i \subset C_i$ be two subsets of $C_i$, $i=1,2$ such that $\text{Cl}(C_i \setminus S_i) = C_i$. Suppose that there exists a homeomorphism $h(C_i \setminus S_i) = C_2 \setminus S_2$ and let $p \in C_i \setminus S_i$ be a limit point of $S_i$. Then the point $h(p) = q$ is a limit point of $S_2$.

Proof. Suppose that $q$ is not a limit point of $S_2$. Since $C_2$ is $0$-dimensional, there exists a closed and open (in $C_2$) neighbourhood $U \subset C_2$ of $q$ such that $U \cap S_2 = \emptyset$. $U$ being closed in $C_2$ it is compact; and since $h^{-1}$ is continuous $h^{-1}(U)$ is also a compact subset of $C_1 \setminus S_1$. But $h^{-1}(U) \subset C_1 \setminus S_1$ is also a neighbourhood of $p$, and since $\text{Cl}(C_1 \setminus S_1) = C_1$ and $p$ is a limit point of $S_1$, there exists a point $p' \in S_1$ such that $p' \in h^{-1}(U)$, which is impossible.

As a trivial consequence of Lemma 2 we obtain the following:

Lemma 3. Let $C_1$ and $C_2$ be two perfect, $0$-dimensional sets (containing more than one point) and let $S_i \subset C_i$ be two subsets of $C_i$ such that $S_1$ is denumerable. Suppose that there exists a homeomorphism $h(C_i \setminus S_i) = C_2 \setminus S_2$ and let $p \in C_i \setminus S_i$ be a limit point of $S_i$, then the point $h(p) = q$ is a limit point of $S_2$.

Indeed, since $S_1$ is denumerable we have $\text{Cl}(C_i \setminus S_i) = C_i$. The other assumptions of Lemma 2 being trivially satisfied it remains to apply this lemma.

Proof of Theorem 3. Since every subset of $C$ which is a $G_\delta$ set is defined by a sequence of open sets and the power of all open subsets of $C$ is $2^{\aleph_0}$, the power of all $G_\delta$ sets does not exceed $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. Therefore it remains to construct a family of power $2^{\aleph_0}$ of $G_\delta$ sets which are dense in themselves and such that no two sets of this family are homeomorphic. We proceed to do this.

Take a perfect subset $P$ of the set $C$ which is nowhere dense in $C$. By Theorem 2 there exists a family $\mathfrak{F}$ of power $2^{\aleph_0}$ of closed subsets of $P$ such that every two sets of $\mathfrak{F}$ are not homeomorphic. Since $P$ is closed and nowhere dense in $C$ the sets of $\mathfrak{F}$ are nowhere dense closed subsets of $C$. Thus for every set $F \in \mathfrak{F}$ there exists a sequence $S \subset C$ of points such that $F \subset C \setminus S$ and $S = F \cup S$. Now take two sets $F_1$ and $F_2$ of $\mathfrak{F}$ and two sequences $S_1$ and $S_2$ of points such that $S_i \subset C$, $F_1 \subset C \setminus S_i$ and $S_i = F_i \cup S_i$.

Consider the sets $C \setminus S_i$, $i = 1, 2$. We shall show that these sets are not homeomorphic. Indeed, suppose that there exists a homeomorphism $h(C \setminus S_i) = C \setminus S_2$. Since $S_i$ is denumerable and $C$ is perfect the assumptions of Lemma 3 hold for $C_i = C_2 = C$. Thus, by $F_i \subset C \setminus S_i$ and $S_i = F_i \cup S_i$ every point $p$ of $F_1$ has an image $h(p)$ in $F_2$ and conversely

\footnote{Evidently $P$ is homeomorphic to $C$.}
for every \( q \in F_2 \) there is \( h^{-1}(q) \in F_1 \). Hence by \( h(C \setminus S_i) = C \setminus S_2 \) there is \( h(F_i) = F_2 \) which is impossible by \( F_i \in \mathcal{F}, i = 1, 2 \).

Thus we can correspond to every set \( F \in \mathcal{F} \) a set \( C \setminus S \), where \( S \) is denumerable, in such a way that the sets \( C \setminus S_1 \) and \( C \setminus S_2 \) corresponding to different sets \( F_1 \) and \( F_2 \) of \( \mathcal{F} \), are not homeomorphic. Since the power of \( \mathcal{F} \) is \( 2^{\aleph_0} \), the power of the family of corresponding sets of the form \( C \setminus S \) is also \( 2^{\aleph_0} \). Since \( S \) is denumerable the sets \( C \setminus S \) are \( G_\delta \) sets and since \( C \) is perfect they are also dense in themselves. Hence Theorem 3 holds.

References