THE POWER OF TOPOLOGICAL TYPES OF SOME CLASSES OF 0-DIMENSIONAL SETS

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By a result of Mazurkiewicz and Sierpinski, there exist $\aleph_1$ topological types of compact and countable sets.[1] Since a countable set is 0-dimensional, there arises a natural question: what is the power of topological types of other classes of 0-dimensional sets? In this paper we consider separable metric spaces only. Every 0-dimensional space being topologically contained in the Cantor set[2] C, we confine ourselves to subsets of this set.

We prove the following three theorems:

THEOREM 1. *There exist two topological types of open subsets of the Cantor set C.*

THEOREM 2. *There exist $2^{\aleph_0}$ topological types of closed subsets of the Cantor set C.*

THEOREM 3. *There exist $2^{\aleph_0}$ topological types of 0-dimensional G_δ sets which are dense in themselves.*

Theorem 1 is known in part,[3] but it seems to the author that an exact proof of it has not been published so far.

Theorems 2 and 3 are new; the latter gives an answer to a problem by Knaster and Urbanik.[4]

The paper contains also some lemmas on homeomorphisms and a notion of a rank $r_p(B)$ of a point p relative to the set B.

1. In this section a lemma on homeomorphisms and the above Theorem 1 are proved.

LEMMA 1. *Let $\{F_n\}$ and $\{G_n\}$ be two sequences of sets satisfying*

(1) $F_n \cap F_m = 0 = G_n \cap G_m$ *for* $n \neq m$,

(2) *for every n the set F_n is open in the union $F = \bigcup_{n=1}^{\infty} F_n$ and G_n is open in $G = \bigcup_{n=1}^{\infty} G_n$, and*

(3) *for every n there exists a homeomorphism h_n such that $h_n(F_n) = G_n$, $n = 1, 2, \cdots$.*

Then the mapping h defined by $h(x) = h_n(x)$ for $x \in F_n$ is a homeomorphism between F and G.

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[1] See [6, p. 22].

[2] See [4, p. 173].

[3] Some general hints may be found in [3, p. 198].

[4] See [3, p. 198].

Proof. By (1) and (3), h is a one-to-one mapping of F onto G. Since the proofs of the continuity of h and h^{-1} are symmetric, we shall show that h is continuous.

Indeed, let $\{x_n\}$ be a sequence of points belonging to F, tending to a point x of F: $x_n \to x$. Since $x \in F$, there exists a number n_0 such that $x \in F_{n_0}$. Now by (2) there exists a number N such that for $n > N$ there is $x_n \in F_{n_0}$ (since otherwise the set F_{n_0} would not be open in F). But h_{n_0} is continuous—as a homeomorphism—and therefore for $n > N$: $h(x_n) = h_{n_0}(x_n) \to h_{n_0}(x) = h(x)$.

Remark 1. Let F_n be the plane set defined by $F_n = \{(x, y); x = 1/n, 0 \leqq y \leqq 1\}$ and put $G_1 = \{(x, y); x = 0; 0 \leqq y \leqq 1\}$ and $G_{n+1} = F_n$, $n = 1, 2, \cdots$. For these sets the assumption (2) of the lemma is not satisfied for the set G_1 only and evidently $F = \bigcup_{n=1}^{\infty} F_n$ is not homeomorphic with $G = \bigcup_{n=1}^{\infty} G_n$, since G is a compact set and F is not. This shows also that assumption (2) of the lemma cannot be replaced by the assumption that F_n and G_n are compact and $\rho(F_n, F_m)$[5] and $\rho(G_n, G_m)$ are positive for all $n \neq m$.

To prove Theorem 1 it suffices to show that:

Every open subset of the Cantor set C is either homeomorphic to C or to C without the zero point: $C \setminus (0)$.

Proof. Let G be an open subset of the Cantor set C. Then G can be written in the form:[6]

(4) $G = G_1 \cup G_2 \cup \cdots$, $G_n \cap G_m = 0$ for $n \neq m$, where the sets G_n are closed and open in C.

Now two cases are possible:

(a) G is a finite union of the sets G_n, i.e. there exists an integer N such that $G_n = 0$ for $n > N$, and

(b) all the sets G_n in (4) are nonempty.

Since

(5) a closed and open subset of the Cantor set C is a perfect set, we see that in case (a) the set G is a perfect 0-dimensional set and therefore homeomorphic to the Cantor set C.

In case (b) we can write the set $C \setminus (0)$ analogically as in (4) in the form:

(6) $C \setminus (0) = F_1 \cup F_2 \cdots$, $F_n \cap F_m = 0$, for $n \neq m$, where the sets F_n are nonempty and closed and open in C.

By (5) there exists for every n a homeomorphism h_n between F_n and G_n and therefore by (4) and (6) the assumptions of the lemma hold.

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[5] By $\rho(F_n, F_m)$ we understand the distance between the sets F_n and F_m, i.e. $\rho(F_n, F_m) = \inf_{x \in F_n, y \in F_m} \rho(x, y)$, where $\rho(x, y)$ denotes the distance between the points x and y.

[6] See [4, p. 166].

Thus by the lemma the set G is, in case (b), homeomorphic to $C\setminus(0)$.

REMARK 2. Theorem 1 may also be proved in another way by using the one-point compactification theorem,[7] but such an exact proof is not simpler than ours.

2. We show in this section that there exist $2^{\aleph_0}$ topological types of closed subsets of the Cantor set C. Since the power of all closed subsets of C is $2^{\aleph_0}$ and every 0-dimensional space has a topological image in the Cantor set C, it suffices to construct a family $\mathfrak{F}$ of power $2^{\aleph_0}$ of compact, 0-dimensional sets, such that no two sets belonging to this family are homeomorphic. To do this we introduce the notion of a rank $r_p(B)$ of a point p relative to the set B. First we recall the notion of the coherence and adherence of a set E in the sense of Hausdorff.[8]

The 0th coherence of E is equal to E; the αth coherence of E is the set of all limits $x=\lim_{n\to\infty} x_n$; $x_i\neq x_j$ for $i\neq j$ such that x and x_n belong to the $(\alpha-1)$th coherence, if $\alpha-1$ exists, and the intersection of all coherences with indices $<\alpha$ if α is a limit number. The αth adherence is the difference between the αth and the $(\alpha+1)$th coherences.

Evidently, the αth adherence is an isolated set. The αth adherence of the set E will be denoted by $E_{(\alpha)}$. It is clear that if E is a compact and countable set and $E^{(\beta)}$ is the last derivative[9] $(\neq 0)$ of E, then $E_{(\beta)}=E^{(\beta)}$ and $E=\bigcup_{\xi\leq\beta} E_{(\xi)}$.

EXAMPLE 1. Take on the x-axis the sets of points defined by: $E_1=\{x;x=1/n, n=1,2,\cdots\}$, $E_2=\{x;x=1/n+1/m, m,n=1,2,\cdots\}$, $E_3=(E_2\setminus E_1)\cup(0)$. Then, the first coherence of E_1 is empty and the first derivative of E_1 consists of the point $x=0$. The first coherence of the set E_3 consists of the point 0. The second coherence of E_3 is empty. The first derivative of E_3 is the set $E_1\cup(0)$ and the second derivative of E_3 consists of the point 0.

We define now the rank $r_p(B)$ of a point $p\in\overline{B}$,[10] where B is a countable set such that $\overline{B}$ is 0-dimensional.[11]

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[7] See [1, p. 93; 5, p. 50].

[8] See [2, p. 132].

[9] The derivative E' of the set E is defined as the set of points $x=\lim_{i\to\infty} x_i$, where $x_i\in E$, $x_i\neq x_j$ for $i\neq j$. Thus in the definition of the derivative there is no need for the point x to belong to E. The αth derivative $E^{(\alpha)}$ is defined as follows: $E^{(0)}=E$; $E^{(\alpha)}=[E^{(\alpha-1)}]'$ if $\alpha-1$ exists and $E^{(\alpha)}=\bigcap_{\xi<\alpha} E^{(\xi)}$ if α is a limit number. If β is the smallest ordinal such that $E^{(\beta)}=E^{(\beta+1)}\neq 0$ or $E^{(\beta+1)}=0$ and $E^{(\beta)}\neq 0$ then $E^{(\beta)}$ is called the last derivative of E and β the order of the last derivative of E.

[10] $\overline{B}$ and $\mathrm{Cl}(B)$ denote the closure of B.

[11] The rank $r_p(B)$ can be defined in a more general case, but for our purpose the above definition is sufficient.

DEFINITION. Let $p \in \overline{B}$ where B is a countable set and $\overline{B}$ is 0-dimensional. If $p \in B_{(0)}$ we define $r_p(B)=0$. If there exists an α such that $p=\lim_{n\to\infty} p_n$ where $p_n \in B_{(\alpha)}$ and p is not a limit point[12] of $B_{(\alpha+1)}$, we define $r_p(B)=\alpha+1$.

If such an α does not exist, then there exist an ordinal α', a sequence $\{\alpha_n'\}$ of ordinals such that $\alpha_n' \to \alpha'$ and a sequence of points $p_n \in B_{(\alpha_n')}$ such that $p=\lim_{n\to\infty} p_n$ and p is not a limit point of $B_{(\alpha')}$. In this case we define $r_p(B)=\alpha'$.

EXAMPLE 2. If E_3 is the set defined in Example 1, the rank of the point 0 relative to E_3 is equal to 1.

Let now E_1 and E_2 be compact and countable sets, such that the ωth derivative $E_1^{(\omega)}$ of E_1 consists of the point p: $E_1^{(\omega)}=(p)$ and the second derivative $E_2^{(2)}$ of E_2 consists of the point q: $E_2^{(2)}=(q)$. Put $E_3=E_1\times(q)\cup(p)\times E_2$ and $B=[(p)\times E_2]\setminus(p, q)$.[13] Then $r_{(p,q)}(B)=2$ and $r_{(p,q)}(E_3)=\omega$.

To define the family $\mathfrak{F}$ a few additional simple remarks are needed.

Since the order α of a coherence is an invariant of homeomorphisms, it is easily seen that

(7) the rank $r_p(B)$ is an invariant of homeomorphisms defined on $\overline{B}$.

Take now the Cantor set C and let E be a compact and countable subset of C such that the ωth derivative $E^{(\omega)}$ of E consists of the point q: $E^{(\omega)}=(q)$. Take the nth adherence $E_{(n)}$ of E, $n=1, 2, \cdots$ and choose from every $E_{(n)}$ a point p_n.

Since the order of an adherence is invariant under homeomorphisms we have that

(8) if h is any homeomorphism of E into itself, then $h(p_n)\neq p_m$ for $n\neq m$.

Let now D_n be the sequence of intervals in the plane defined by $D_n=\{(x, y);\ x=p_n,\ 0\leqq y\leqq 1\}$ $n=1, 2, \cdots$ and let $\{\alpha_n\}$ be a sequence of ordinals: $1<\alpha_n<\Omega$. Choose in every D_n a countable and compact subset F_n such that α_n be the order of the last derivative $F_n^{(\alpha_n)}$ of F_n and that $F_n^{(\alpha n)}=(p_n)$. Then the set $A=C\cup\bigcup_{n=1}^{\infty} F_n$ is compact (since the diameters of D_n are equal to $1/n$ and $F_n\subset D_n$) and 0-dimensional. By the definition of F_n we have also

$$r_{p_n}\left(\bigcup_{n=1}^{\infty} F_n\setminus E\right)=\alpha_n>1, \qquad n=1, 2, \cdots. \tag{9}$$

Now take in the plane an arbitrary bounded and isolated set I

[12] A point x such that there exists a sequence $\{x_n\}$ of points x_n belonging to E, $x_n\neq x_m$ for $n\neq m$ and such that $x_n\to x$ is called a limit point of E.

[13] $\times$ denotes the Cartesian product and (p, q) is the point in the Cartesian product.

disjoint with C such that $I^{(1)}=E$. Then the set $A_1=C\cup\bigcup_{n=1}^{\infty} F_n\cup I$ is 0-dimensional and compact. Denoting the decomposition of A_1 according to the theorem of Cantor-Bendixson by $A_1=P_1\cup B_1$ with P_1 as perfect set, we obtain

$$P_1 = C \quad \text{and} \quad B_1 = \left(\bigcup_{n=1}^{\infty} F_n \cup I\right)\backslash E$$

and by the definition of I,

$$P_1 \cap \overline{B}_1 = E.$$

Since I is isolated there is also, by (9),

(10) $r_{p_n}(B_1)=\alpha_n>1$ and for every $p\in E$ and $p\neq p_n$, $r_p(B_1)=1$.

If we take now any other sequence $\{\beta_n\}$ of ordinals: $1<\beta_n<\Omega$ and the same set E and points p_n, we can construct, analogically as before, a 0-dimensional and compact set A_2 with the following properties:

If we denote the decomposition of A_2 according to the theorem of Cantor-Bendixson by $A_2=P_2\cup B_2$ with P_2 as perfect set, then $P_2=C$ and

$$P_2 \cap \overline{B}_2 = E.$$

Also

(10′) $r_{p_n}(B_2)=\beta_n>1$ and for every $p\in E$ and $p\neq p_n$, $r_p(B_2)=1$.

Now suppose that there exists a homeomorphism h between A_1 and A_2: $h(A_1)=A_2$. Then we would have $h(P_1\cap\overline{B}_1)=P_2\cap\overline{B}_2$, i.e., $h(E)=E$. Hence by (8) there would be $h(p_n)\neq p_m$ for $n\neq m$. But, by (7), (10) and (10′) there must be $h(p_n)=p_n$ and therefore by (7), $\alpha_n=\beta_n$ for every n. This shows that if the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are different, the sets A_1 and A_2 cannot be homeomorphic. But the power of all sequences $\{\alpha_n\}$, $1<\alpha_n<\Omega$ is $\aleph_1^{\aleph_0}=2^{\aleph_0}$. Hence Theorem 2 holds.

Remark 3. In [7, p. 119], we introduced a function $\sigma_A(B)$ assigning to every 0-dimensional compact set A an ordinal $<\Omega$. Using this function, it can be easily shown that the power of all topological types of compact uncountable subsets of the Cantor set is $\aleph_1$. (This can be also obtained from the result of Mazurkiewicz and Sierpinski, mentioned at the beginning of this paper.) Thus by the continuum hypothesis it is equal to $2^{\aleph_0}$, but we proved this fact without recourse to this hypothesis.

Note also that the fact that there exist $2^{\aleph_0}$ topological types of closed sets (not necessarily 0-dimensional) was stated in [6, p. 27].

3. In this section a proof of Theorem 3 is given. Two lemmas are also proved.

LEMMA 2. *Let C_1 and C_2 be two compact* 0-*dimensional sets and let $S_i \subset C_i$ be two subsets of C_i, $i=1, 2$ such that* $\mathrm{Cl}(C_1 \setminus S_1) = C_1$. *Suppose that there exists a homeomorphism $h(C_1 \setminus S_1) = C_2 \setminus S_2$ and let $p \in C_1 \setminus S_1$ be a limit point of S_1. Then the point $h(p) = q$ is a limit point of S_2.*

PROOF. Suppose that q is not a limit point of S_2. Since C_2 is 0-dimensional, there exists a closed and open (in C_2) neighbourhood $U \subset C_2$ of q such that $U \cap S_2 = 0$. U being closed in C_2 it is compact; and since h^{-1} is continuous $h^{-1}(U)$ is also a compact subset of $C_1 \setminus S_1$. But $h^{-1}(U) \subset C_1 \setminus S_1$ is also a neighbourhood of p, and since $\mathrm{Cl}(C_1 \setminus S_1) = C_1$ and p is a limit point of S_1, there exists a point $p' \in S_1$ such that $p' \in h^{-1}(U)$, which is impossible.

As a trivial consequence of Lemma 2 we obtain the following:

LEMMA 3. *Let C_1 and C_2 be two perfect,* 0-*dimensional sets (containing more than one point) and let $S_i \subset C_i$ be two subsets of C_i such that S_1 is denumerable. Suppose that there exists a homeomorphism $h(C_1 \setminus S_1) = C_2 \setminus S_2$ and let $p \in C_1 \setminus S_1$ be a limit point of S_1, then the point $h(p) = q$ is a limit point of S_2.*

Indeed, since S_1 is denumerable we have $\mathrm{Cl}(C_1 \setminus S_1) = C_1$. The other assumptions of Lemma 2 being trivially satisfied it remains to apply this lemma.

PROOF OF THEOREM 3. Since every subset of C which is a G_δ set is defined by a sequence of open sets and the power of all open subsets of C is $2^{\aleph_0}$, the power of all G_δ sets does not exceed $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. Therefore it remains to construct a family of power $2^{\aleph_0}$ of G_δ sets which are dense in themselves and such that no two sets of this family are homeomorphic. We proceed to do this.

Take a perfect subset P of the set C which is nowhere dense in C. By Theorem 2 there exists a family $\mathfrak{F}$ of power $2^{\aleph_0}$ of closed subsets of P[14] such that every two sets of $\mathfrak{F}$ are not homeomorphic. Since P is closed and nowhere dense in C the sets of $\mathfrak{F}$ are nowhere dense closed subsets of C. Thus for every set $F \in \mathfrak{F}$ there exists a sequence $S \subset C$ of points such that $F \subset C \setminus S$ and $\overline{S} = F \cup S$. Now take two sets F_1 and F_2 of $\mathfrak{F}$ and two sequences S_1 and S_2 of points such that $S_i \subset C$, $F_i \subset C_i \setminus S_i$ and $\overline{S}_i = F_i \cup S_i$.

Consider the sets $C \setminus S_i$, $i=1, 2$. *We shall show that these sets are not homeomorphic.* Indeed, suppose that there exists a homeomorphism $h(C \setminus S_1) = C \setminus S_2$. Since S_1 is denumerable and C is perfect the assumptions of Lemma 3 hold for $C_1 = C_2 = C$. Thus, by $F_i \subset C \setminus S_i$ and $\overline{S}_i = F_i \cup S_i$ every point p of F_1 has an image $h(p)$ in F_2 and conversely

[14] Evidently P is homeomorphic to C.

for every $q \in F_2$ there is $h^{-1}(q) \in F_1$. Hence by $h(C \setminus S_1) = C \setminus S_2$ there is $h(F_1) = F_2$ which is impossible by $F_i \in \mathfrak{F}$, $i = 1, 2$.

Thus we can correspond to every set $F \in \mathfrak{F}$ a set $C \setminus S$, where S is denumerable, in such a way that the sets $C \setminus S_1$ and $C \setminus S_2$, corresponding to different sets F_1 and F_2 of $\mathfrak{F}$, are not homeomorphic. Since the power of $\mathfrak{F}$ is $2^{\aleph_0}$ the power of the family of corresponding sets of the form $C \setminus S$ is also $2^{\aleph_0}$. Since S is denumerable the sets $C \setminus S$ are G_δ sets and since C is perfect they are also dense in themselves. Hence Theorem 3 holds.

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