FIXED POINT THEOREMS FOR PSEUDO MONOTONE MAPPINGS

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1. Introduction. Recently [7] the author generalized a well-known theorem of Hamilton [1] in the following manner: if $X$ is a continuum each of whose subcontinua is unicoherent and decomposable, then $X$ has the fixed point property for monotone transformations. As a corollary it followed that the same fixed point property obtains for continua each of whose nondegenerate subcontinua has a cutpoint. The argument depended on the order structure of a certain arcwise connected hyperspace of the continuum.

In this note we arrive at the same corollary by a distinctly different and simpler proof. Au fond the argument is essentially the same as one due to Kelley [2] where it was shown that a homeomorphism of a continuum into itself has an invariant, cutpoint-free subcontinuum. (The analogous result for monotone transformations was proved by the author in [6].) The proof of Kelley does not make full use of the properties of homeomorphisms; the essential properties which make his argument work define a class of transformations which we shall term the pseudo monotone mappings.

Finally, we note that our results for pseudo monotone mappings admit a further generalization in the setting of partially ordered topological spaces.

2. Pseudo monotone mappings. Let $X$ and $Y$ be spaces and $f: X \to Y$ a continuous mapping. We say that $f$ is pseudo monotone if, whenever $A$ and $B$ are closed and connected subsets of $X$ and $Y$, respectively, and $B \subseteq f(A)$, it follows that some component of $A \cap f^{-1}(B)$ is mapped by $f$ onto $B$. In general this notion is independent of that of a monotone mapping, but in certain applications of interest every monotone mapping is pseudo monotone.

Recall that a continuum (= compact connected Hausdorff space) is hereditarily unicoherent if any two of its subcontinua meet in a connected set.

Lemma 1. If $X$ is an hereditarily unicoherent continuum and $f: X \to Y$ is a monotone mapping, then $f$ is pseudo monotone.

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Proof. Let \( A \) and \( B \) be closed and connected subsets of \( X \) and \( Y \), respectively, such that \( B \subseteq f(A) \). Since \( f \) is monotone, \( f^{-1}(B) \) is a continuum, and since \( X \) is hereditarily unicoherent, \( A \cap f^{-1}(B) \) is connected. Hence \( f \) is pseudo monotone.

Suppose now that \( X \) is a continuum and that \( f: X \to X \) is continuous. A simple maximality argument establishes the existence of a nonempty subcontinuum \( Y \), which is minimal with respect to being invariant under \( f \). Suppose \( Y \) has a cutpoint \( p \), with

\[
Y - p = A \cup B
\]

where \( A \) and \( B \) are disjoint, separated and nonempty. If \( f(p) = p \) then the minimality of \( Y \) is contradicted, so we may assume \( f(p) \in A \) and define \( r(Y) = \overline{A} \) by

\[
r(x) = \begin{cases} 
  x, & x \in \overline{A}, \\
  p, & x \in \overline{B}.
\end{cases}
\]

The mapping \( g: \overline{A} \to \overline{A} \) defined by \( g = rf \) is continuous, and the set

\[
K = \bigcap_{n=1}^{\infty} \{ g^n(\overline{A}) \}
\]

is a subcontinuum of \( \overline{A} \) which is invariant under \( g \). Thus

\[
f(K) \cap K = rf(K) = g(K) = K
\]

and we infer \( K \subseteq f(K) \). Therefore, if \( f \) is pseudo monotone, the set \( K \cap f^{-1}(K) \) has a component \( K_1 \) such that \( f(K_1) = K \). Inductively we obtain a sequence of subcontinua, \( K_n \), such that

\[
K_n \subset f(K_n) = K_{n-1} \subset \cdots \subset f(K_1) = K.
\]

Clearly, the intersection of this sequence is a nonempty subcontinuum invariant under \( f \), and this contradicts the minimality of \( Y \). We have proved

**Theorem 1.** If \( X \) is a continuum and \( f: X \to X \) is a pseudo monotone mapping, then \( X \) contains a nonempty subcontinuum \( Y \) which is minimal with respect to being invariant under \( f \). Moreover, \( Y \) has no cutpoints.

**Corollary 1.1.** If \( X \) is a continuum such that each of its nondegenerate subcontinua has a cutpoint, and if \( f: X \to X \) is a pseudo monotone mapping, then there exists \( x_0 \in X \) such that \( x_0 = f(x_0) \).

It has been proved elsewhere [7] that the continua of Corollary 1.1 are hereditarily unicoherent. Therefore, by Lemma 1, we have
Corollary 1.2. If $X$ is a continuum such that each of its nondegenerate subcontinua has a cutpoint, and if $f: X \rightarrow X$ is a monotone mapping, then there exists $x_0 \in X$ such that $x_0 = f(x_0)$.

3. A generalization. In [5] the author defined a POTS (= partially ordered topological space) to be a partially ordered set $X$, so topologized that the sets

$$L(x) = \{a: a \leq x\}, \quad M(x) = \{a: x \leq a\}$$

are closed, for each $x \in X$. Two elements $x$ and $y$ of $X$ are comparable if $x \leq y$ or $y \leq x$. In the event $X$ contains a unit, i.e., a unique element $e$ such that $L(e) = X$, we say that the subset $A$ of $X$ is bounded away from $e$ if there exists $y \neq e$ such that $A \subseteq L(y)$.

The following theorem was proved in [5].

**Fixed Point Theorem.** Let $X$ be a compact Hausdorff POTS with unit, $e$. Let $f: X \rightarrow X$ be a continuous, order-preserving mapping satisfying the following conditions.

(i) There exists $x \neq e$ such that $x$ and $f(x)$ are comparable.

(ii) If $x \neq e$ and if $x$ and $f(x)$ are comparable, then either the sequence $f^n(x), n = 1, 2, \ldots$, is bounded away from $e$, or $f^{-1}(x) \cap L(x)$ is nonempty.

Then there exists $x_0 \neq e$ such that $x_0 = f(x_0)$.

For the remainder of this paper let us assume that $X$ is a compact Hausdorff POTS with unit $e$, which is endowed with the following two properties.

(a) There exist elements $a$, $b$ and $p$ of $X$ such that $L(a) \cap L(b) = p$.

(b) If $x \in X - L(a) \cup L(b)$ then $p \leq x$ and each of the sets $L(x) \cap L(a)$ and $L(x) \cap L(b)$ has a supremum.

Let $f: X \rightarrow X$ be continuous and order-preserving, and suppose $f$ maps minimal elements into minimal elements. In addition, suppose $f$ satisfies the following order-theoretic analogue of pseudo monotonicity.

(P) If $x \leq f(x)$ then $f^{-1}(x) \cap L(x)$ is nonempty.

According to the fixed point theorem above, $f$ has a fixed point distinct from $e$ if $f(x) \leq x$ for some $x \neq e$. If this does not occur, then by (b) and the fact that $f(p)$ is minimal, we have $f(p) \leq a$ or $f(p) \leq b$, but not both. Suppose $f(p) \leq a$; since $f$ is order-preserving, $f(a)$ cannot lie in $L(b)$. Moreover, $f(a)$ cannot lie in $L(a)$ by assumption, so that by (b) there must exist

$$t_1 = \sup \{L(f(a)) \cap L(a)\},$$
with $p \leq t_1$. Now $f(t_1) \in X - L(a)$ and, since $p \leq t_1$, it follows that $f(p) \leq f(t_1)$ and hence $f(t_1) \in X - L(b)$. Applying (β) again there exists
\[ t_2 = \sup(L(f(t_1)) \cap L(a)) \]
with $p \leq t_2$. Because $f$ is order-preserving it follows that $f(t_1) \leq f(a)$ and hence $t_2 \leq f(a)$. Moreover, $t_2 \leq a$ so that $t_2 \leq t_1$. Inductively, we obtain a sequence $t_n$ satisfying
\[ t_{n+1} = \sup(L(f(t_n)) \cap L(a)), p \leq t_{n+1} \leq t_n. \]
Since $t_n$ is a decreasing sequence, it must converge to some $t_0 \leq t_n$. Further, since $t_n \leq f(t_{n-1})$, it follows that $t_0 \leq f(t_0)$. Condition (i) is now satisfied and (ii) follows from (P) and the above discussion. Hence we infer (compare with a result of A. D. Wallace [4])

**Theorem 2.** Let $X$ be a nondegenerate compact Hausdorff POTS with unit $e$, satisfying (α) and (β). Let $f: X \rightarrow X$ be a continuous, order-preserving mapping which maps minimal elements into minimal elements and satisfies (P). Then there exists $x_0 \in X - e$ such that $f(x_0) = x_0$.

It is not difficult to see that Theorem 2 is truly a generalization of Theorem 1. Let $Y$ be a continuum with a cutpoint $p$, and let $f(Y) = Y$ be pseudo monotone. If $X$ is the space of subcontinua of $Y$, endowed with the finite topology [3], and if $f^*$ is the mapping of $X$ into itself induced by $f$, then $X$ and $f^*$ satisfy the hypotheses of Theorem 2, where the partial order is taken to be inclusion. Thus $Y$ contains an invariant proper subcontinuum and Theorem 1 follows.

**References**


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