A REMARK ON GENERALISED FREE PRODUCTS[1]

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The aim of this note is to prove the following fact, which does not seem to have been noticed before: The generalised free product of two finitely presented groups is finitely presented if and only if the amalgamated subgroup is finitely generated.

The one part of the theorem is immediately obvious. For the other, let us suppose that

$$A = gp(a_1, a_2, \cdots, a_u; r_1 = 1, r_2 = 1, \cdots, r_k = 1)$$

and

$$B = gp(b_1, b_2, \cdots, b_v; s_1 = 1, s_2 = 1, \cdots, s_m = 1)$$

are the given finitely presented groups. Suppose that the subgroups to be amalgamated are H_1 and H_2. Let

$$H_i = gp(h_{ij}; j = 1, 2, \cdots) \qquad (i = 1, 2)$$

where the mapping

$$h_{1j} \rightarrow h_{2j} \qquad (j = 1, 2, \cdots)$$

defines the identifying isomorphism in the generalised free product G of A and B. We may, by the very definition of the generalized free product (see B. H. Neumann [1]), define G as a factor group F/R of the free group F on $a_1, a_2, \cdots, a_u, b_1, b_2, \cdots, b_v$ by the smallest normal subgroup R of F containing

$$(1) \qquad r_1, r_2, \cdots, r_k, s_1, s_2, \cdots, s_m, h_{11}h_{21}^{-1}, h_{12}h_{22}^{-1}, h_{13}h_{23}^{-1}, \cdots.$$

By a well-known theorem of Neumann [2], if G can be finitely defined, then R is in fact the normal closure of only finitely many of the elements given in (1). Let us suppose that this is indeed the case. Thus we may suppose that R is the normal closure of

$$(2) \qquad r_1, r_2, \cdots, r_k, s_1, \cdots, s_m, h_{11}h_{21}^{-1}, h_{12}h_{22}^{-1}, \cdots, h_{1n}h_{2n}^{-1}.$$

We remind the reader that we have assumed that H_1 is not finitely generated. Thus

$$H_1' = gp(h_{11}, h_{12}, \cdots, h_{1n})$$

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is a proper subgroup of H_1. Thus there exists a positive integer w, say, such that $h_{1w} \notin H_1'$.

Now let

$$H_2' = gp(h_{21}, h_{22}, \cdots, h_{2n}).$$

Form G' the generalised free product of A and B amalgamating H_1' and H_2':

$$G' = \{A * B; H_1' = H_2'\}.$$

We now map F onto G' by mapping a_i, qua element of F, onto the element a_i of G' and similarly for the b_j. Observe then that the kernel of this homomorphism contains all the elements in (2) and so coincides with R. Now

$$h_{1w}h_{2w}^{-1} \in R;$$

or, to put it another way, $h_{1w}h_{2w}^{-1}$, qua element of G', is the identity. But $h_{1w}h_{2w}^{-1}$ represents an element of length 2 in G' since

$$h_{1w} \in A, \qquad h_{1w} \notin H_1'$$

and

$$h_{2w} \in B, \qquad h_{2w} \notin H_2'$$

(see Neumann [1, Corollary 3.2]). Thus $h_{1w}h_{2w}^{-1}$ is certainly not the identity element of G' and so we have arrived at a contradiction. Therefore G cannot be defined by a finite number of relations and the proof of the theorem is complete.

It is possible that this simple theorem may be of use in logic. For example, one can prove easily thereby that there is no effective procedure to determine whether any group, which is defined by a finite number of generators and a recursive set of defining relations, is finitely presented (see Baumslag [3] for a different proof of this result and for an explanation of the terms used here).

References

1. B. H. Neumann, *An essay on free products of groups with amalgamations*, Philos. Trans. Roy. Soc. London. Ser. A. **246** (1954), 503–554.

2. ———, *Some remarks on infinite groups*, J. London Math. Soc. **12** (1937), 120–127.

3. G. Baumslag, *Wreath products and finitely presented groups*, Math. Z. **75** (1961), 22–28.

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