1. Notations and terminology. Our terminology conforms with that of [2]. The inner product of vectors $x$ and $y$ in a Hilbert space $\mathcal{H}$ is denoted $(x, y)$. An operator in $\mathcal{H}$ is a continuous linear mapping $T: \mathcal{H} \to \mathcal{H}$. The *-algebra of all operators in $\mathcal{H}$ is denoted $L(\mathcal{H})$. A complex number $\mu$ is a proper value for $T$ if there exists a nonzero vector $x$ such that $(T - \mu I)x = 0$; such a vector $x$ is a proper vector for $T$. A complex number $\mu$ is an approximate proper value for $T$ in case there exists a sequence of vectors $x^n$ such that $\|x^n\| = 1$ and $\|Tx^n - \mu x^n\| \to 0$; equivalently, there does not exist a number $\epsilon > 0$ such that $(T - \mu I)^*(T - \mu I) \geq \epsilon I$.

The spectrum of an operator $T$, denoted $\sigma(T)$, is the set of all complex numbers $\mu$ such that $T - \mu I$ has no inverse. The approximate point spectrum of $T$, denoted $\sigma_a(T)$, is the set of all approximate proper values of $T$. The point spectrum of $T$, denoted $\sigma_p(T)$, is the set of all proper values of $T$. Evidently $\sigma_p(T) \subset \sigma_a(T) \subset s(T)$. If $T$ is normal, $s(T) = \sigma_a(T)$ (see [2, Theorem 31.2]); if $T$ is Hermitian, $\sigma_a(T)$ contains a (necessarily real) number $\alpha$ such that $|\alpha| = \|T\|$ (see [2, Theorem 34.2]), and in particular one has an elementary proof of the fact that the spectrum of $T$ is nonempty.

2. Introduction. The spectrum of a Hermitian operator is shown to be nonempty by completely elementary means. It would be nice to have an elementary proof for normal operators (see [2, p. 111]). The purpose of this note is to give a proof based on Banach limits. Incidentally, $\mathcal{H}$ will be extended to a curious Hilbert space $\mathcal{X}$, in which it becomes natural to speak of “approximate proper vectors.”

Our motivation for the construction of $\mathcal{X}$ was as follows. Suppose $T$ is a normal operator, and $\mu$ and $\nu$ are distinct approximate proper values of $T$. Choose sequences of unit vectors $\{x_n\}$ and $\{y_n\}$ such that $\|Tx_n - \mu x_n\| \to 0$ and $\|Ty_n - \nu y_n\| \to 0$. Then,

$$\|(\mu - \nu)(x_n, y_n)\| = \|(\mu x_n - Tx_n, y_n) + (x_n, T^*y_n - \nu^*y_n)\| \leq \|\mu x_n - Tx_n\| + \|T^*y_n - \nu^*y_n\| = \|\mu x_n - Tx_n\| + \|Ty_n - \nu y_n\| \to 0.$$

Thus, $(x_n, y_n) \to 0$, and we have a generalization of the following well-
known fact: for a normal operator, proper vectors belonging to distinct proper values are orthogonal. This suggests thinking of the sequences \( \{x_n\} \) and \( \{y_n\} \) as being "approximate proper vectors," with their inner product defined to be \( \lim(x_n, y_n) \).

In what follows, we denote by \( \text{glim} \) a fixed "Banach generalized limit," defined for bounded sequences \( \{\lambda_n\} \) of complex numbers (see page 34 of [1]); thus,

(1) \( \text{glim}(\lambda_n + \mu_n) = \text{glim} \lambda_n + \text{glim} \mu_n \),
(2) \( \text{glim}(\lambda \lambda_n) = \lambda \text{glim} \lambda_n \),
(3) \( \text{glim} \lambda_n = \lim \lambda_n \) whenever \( \{\lambda_n\} \) is convergent,
(4) \( \text{glim} \lambda_n \geq 0 \) when \( \lambda_n \geq 0 \) for all \( n \).

We shall not make use of a "translation-invariant" property of \( \text{glim} \); all we need are properties (1)-(4), in other words, a positive linear form on the vector space \( \mathbb{m} \) of bounded sequences, which vanishes on the space \( \mathcal{c}_0 \) of null sequences, and has the value 1 for the constant sequence \( \{1\} \). It follows from (1) and (4) that \( \text{glim} \lambda_n \) is real whenever \( \lambda_n \) is real for all \( n \); this implies in turn that \( \text{glim}(\lambda_n^*) = (\text{glim} \lambda_n)^* \) for any bounded sequence \( \{\lambda_n\} \).

3. An extension \( \mathcal{K} \) of \( \mathcal{K} \). Denote by \( \mathcal{B} \) the set of all sequences \( s = \{x_n\} \), with \( x_n \) in \( \mathcal{K} \) (\( n = 1, 2, 3, \ldots \)), such that \( \|x_n\| \) is bounded (that is, \( \{\|x_n\|\} \) is in \( \mathcal{m} \)). If \( s = \{x_n\} \) and \( t = \{y_n\} \), write \( s = t \) in case \( x_n = y_n \) for all \( n \). The set \( \mathcal{B} \) is a vector space relative to the definitions \( s + t = \{x_n + y_n\} \) and \( \lambda s = \{\lambda x_n\} \).

Suppose \( s = \{x_n\} \) and \( t = \{y_n\} \) belong to \( \mathcal{B} \); since \( |(x_n, y_n)| \leq \|x_n\| \|y_n\| \), it is permissible to define

\[ \phi(s, t) = \text{glim}(x_n, y_n). \]

Evidently \( \phi \) is a positive symmetric bilinear functional on \( \mathcal{B} \) (see [2, §2]), hence \( |\phi(s, t)| \leq \phi(s, s)\phi(t, t) \) (see [2, §5]). Let \( \mathcal{N} = \{s: \phi(s, s) = 0\} = \{s: \phi(s, t) = 0 \text{ for all } t \in \mathcal{B}\} \). Clearly \( \mathcal{N} \) is a linear subspace of \( \mathcal{B} \); we write \( s' \) for the coset \( s + \mathcal{N} \). The quotient vector space \( \mathcal{Q} = \mathcal{B}/\mathcal{N} \) becomes an inner product space on defining \( (s', t') = \phi(s, t) \). Thus, if \( u = \{x_n\}' \) and \( v = \{y_n\}' \),

\[ (u, v) = \text{glim}(x_n, y_n). \]

If \( x \) is in \( \mathcal{K} \), we write \( \{x\} \) for the sequence all of whose terms are \( x \), and \( x' \) for the coset \( \{x\} + \mathcal{N} \). Evidently \( (x', y') = (x, y) \), and \( x \rightarrow x' \) is an isometric linear mapping of \( \mathcal{K} \) onto a closed linear subspace \( \mathcal{K}' \) of \( \mathcal{Q} \). Regard \( \mathcal{Q} \) as a linear subspace of its Hilbert space completion.
Thus, \( \mathcal{K} \subseteq \mathcal{L} \subseteq \mathcal{K} \), where \( \mathcal{K} \) is a closed linear subspace of \( \mathcal{K} \), and \( \mathcal{L} \) is a dense linear subspace of \( \mathcal{K} \).

4. A representation of \( L(\mathcal{K}) \). Every operator \( T \) in \( \mathcal{K} \) determines an operator \( T^0 \) in \( \mathcal{K} \), as follows.

If \( s = \{ x_n \} \) is in \( \mathcal{B} \), then the relation \( \| Tx_n \| \leq \| T \| \| x_n \| \) shows that \( \{ Tx_n \} \) is in \( \mathcal{B} \). Defining \( T_\mathcal{B} = \{ Tx_n \} \), we have a linear mapping \( T_\mathcal{B} : \mathcal{B} \rightarrow \mathcal{B} \) such that \( \phi (T_\mathcal{B} s, T_\mathcal{B} s) \leq \| T \|^2 \phi (s, s) \). In particular if \( s \) is in \( \mathcal{K} \), that is if \( \phi (s, s) = 0 \), then \( T_\mathcal{B} s \) is also in \( \mathcal{K} \). It follows that \( \{ x_n \}' \rightarrow \{ Tx_n \}' \) is a well-defined linear mapping of \( \mathcal{L} \) into \( \mathcal{L} \), which we denote \( T^0 \); thus, \( T^0 s' = (T_\mathcal{B} s)' \), and the inequality \( (T^0 u, T^0 w) \leq \| T \|^2 (u, w) \), valid for all \( u \) in \( \mathcal{L} \), shows that \( T^0 \) is continuous, with \( \| T^0 \| \leq \| T \| \).

Since in particular \( T^0 x' = (Tx)' \) for all \( x \) in \( \mathcal{K} \), it is clear that \( \| T^0 \| \geq \| T \| \), thus \( \| T^0 \| = \| T \| \). The continuous linear mapping \( T^0 \) extends to a unique operator in \( \mathcal{K} \), which we also denote \( T^0 \).

The mapping \( T \rightarrow T^0 \) of \( L(\mathcal{K}) \) into \( L(\mathcal{K}) \) is easily seen to be a faithful \( \ast \)-representation: \( (S + T)^0 = S^0 + T^0 \), \( (\lambda T)^0 = \lambda T^0 \), \( (ST)^0 = S^0 T^0 \), \( (T^*)^0 = (T^0)^* \), \( I^0 = I \), and \( \| T^0 \| = \| T \| \).

Suppose \( T \geq 0 \), that is, \( (Tx, x) \geq 0 \) for all \( x \) in \( \mathcal{K} \). If \( u = \{ x_n \}' \) is in \( \mathcal{L} \), then \( (Tx_n, x_n) \geq 0 \) for all \( n \), hence \( (T^0 u, u) = \text{glim} \| Tx_n, x_n \| \geq 0 \); it follows that \( T^0 v, v \geq 0 \) for all \( v \) in \( \mathcal{K} \). Clearly: for an operator \( T \) in \( \mathcal{K} \), one has \( T \geq 0 \) if and only if \( T^0 \geq 0 \).

**Lemma.** If \( T \) is any operator in \( \mathcal{K} \), \( a(T^0) = a(T) \).

**Proof.** A complex number \( \mu \) fails to belong to \( a(T) \) if and only if there exists a number \( \epsilon > 0 \) such that \( (T - \mu I)^* (T - \mu I) \geq \epsilon I \). By the above remarks, this condition is equivalent to \( (T^0 - \mu I)^* (T^0 - \mu I) \geq \epsilon I \).

**Theorem 1.** For every operator \( T \) in \( \mathcal{K} \),

\[
a(T) = a(T^0) = \rho(T^0).
\]

**Proof.** The relations \( a(T) = a(T^0) \supseteq \rho(T^0) \) have already been noted. Suppose \( \mu \) is in \( a(T) \). Choose a sequence \( x_n \) in \( \mathcal{K} \) such that \( \| x_n \| = 1 \) and \( \| Tx_n - \mu x_n \| \rightarrow 0 \), and set \( u = \{ x_n \}' \). Clearly \( \| u \| = 1 \) and \( \| T^0 u - \mu u \| \leq \text{glim} \| Tx_n - \mu x_n \|^2 = 0 \), hence \( T^0 u = \mu u \); that is, \( \mu \) is in \( \rho(T^0) \).

**Theorem 2.** If \( T \) is any normal operator in \( \mathcal{K} \), \( T \) has an approximate proper value \( \mu \) such that \( | \mu | = \| T \| \).

**Proof.** Without loss of generality, we may suppose \( \| T \| = 1 \). If \( 1 \) is in \( s(T) \), the relation \( s(T) = a(T) \) ends the proof. Let us assume henceforth that \( I - T \) is invertible.
Let $S = T^*T$. Since $\|S\| = 1$, and since $S \geq 0$, it follows from the remarks in §1 that $1$ is an approximate proper value for $S$. By Theorem 1, $1$ is a proper value for $S^o$. Let $\mathcal{M}$ be the null space of $S^o - I$, thus $\mathcal{M} = \{v : S^ov = v \neq 0\}$. Since $TS = ST$ and $T^*S = ST^*$, $\mathcal{M}$ is invariant under $T^o$ and $(T^o)^*$. Thus, $\mathcal{M}$ reduces $T^o$. We denote by $T^o/\mathcal{M}$ the restriction of $T^o$ to $\mathcal{M}$. Since $S^o/\mathcal{M} = I$, we have $(T^o/\mathcal{M})*(T^o/\mathcal{M}) = (T^o/s/\mathcal{M})(T^o/\mathcal{M}) = (T^o*T^o)/\mathcal{M} = S^o/\mathcal{M} = I$; clearly $T^o/\mathcal{M}$ is a unitary operator in $\mathcal{M}$. Write $U = T^o/\mathcal{M}$. Since $I - T$ has an inverse in $L(\mathcal{H})$, $I - T^o$ has an inverse in $L(\mathcal{H})$; since $\mathcal{M}$ reduces $I - T^o$, it follows that $I - U$ has an inverse in $L(\mathcal{M})$. Let $R$ be the Cayley transform of $U$, that is, $R = i(I + U)(I - U)^{-1}$; $R$ is a Hermitian operator in $\mathcal{M}$. Define $A = i(I + T)(I - T)^{-1}$; clearly $A^o/\mathcal{M} = R$.

Let $\alpha$ be any approximate proper value for $R$ (see §1). It is clear from the definition that $\alpha$ is also an approximate proper value for $A^o$. By Theorem 1, there is a nonzero vector $u$ in $\mathcal{M}$ such that $A^o u = \alpha u$. Since $A^o = i(I + T^o)(I - T^o)^{-1}$, an elementary calculation gives $T^o u = (\alpha - i)(\alpha + i)^{-1} u$. Thus, $\mu = (\alpha - i)(\alpha + i)^{-1}$ belongs to $\rho(T^o) = \alpha(T^o) = A(T^o)$, and $|\mu| = 1 = \|T\|$.

References


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