1. Notations and terminology. Our terminology conforms with that of [2]. The inner product of vectors $x$ and $y$ in a Hilbert space $\mathcal{K}$ is denoted $(x, y)$. An operator in $\mathcal{K}$ is a continuous linear mapping $T: \mathcal{K} \to \mathcal{K}$. The $*$-algebra of all operators in $\mathcal{K}$ is denoted $L(\mathcal{K})$. A complex number $\mu$ is a proper value for $T$ if there exists a nonzero vector $x$ such that $(T-\mu I)x = 0$; such a vector $x$ is a proper vector for $T$. A complex number $\mu$ is an approximate proper value for $T$ in case there exists a sequence of vectors $x_n$ such that $\|x_n\| = 1$ and $\|T x_n - \mu x_n\| \to 0$; equivalently, there does not exist a number $\epsilon > 0$ such that $(T - \mu I)^* (T - \mu I) \geq \epsilon I$.

The spectrum of an operator $T$, denoted $s(T)$, is the set of all complex numbers $\mu$ such that $T - \mu I$ has no inverse. The approximate point spectrum of $T$, denoted $a(T)$, is the set of all approximate proper values of $T$. The point spectrum of $T$, denoted $\rho(T)$, is the set of all proper values of $T$. Evidently $\rho(T) \subseteq a(T) \subseteq s(T)$. If $T$ is normal, $s(T) = a(T)$ (see [2, Theorem 31.2]); if $T$ is Hermitian, $a(T)$ contains a (necessarily real) number $\alpha$ such that $|\alpha| = \|T\|$ (see [2, Theorem 34.2]), and in particular one has an elementary proof of the fact that the spectrum of $T$ is nonempty.

2. Introduction. The spectrum of a Hermitian operator is shown to be nonempty by completely elementary means. It would be nice to have an elementary proof for normal operators (see [2, p. 111]).

The purpose of this note is to give a proof based on Banach limits. Incidentally, $\mathcal{K}$ will be extended to a curious Hilbert space $\hat{\mathcal{K}}$, in which it becomes natural to speak of “approximate proper vectors.”

Our motivation for the construction of $\hat{\mathcal{K}}$ was as follows. Suppose $T$ is a normal operator, and $\mu$ and $\nu$ are distinct approximate proper values of $T$. Choose sequences of unit vectors $\{x_n\}$ and $\{y_n\}$ such that $\|T x_n - \mu x_n\| \to 0$ and $\|T y_n - \nu y_n\| \to 0$. Then,

$$\lim \frac{(\mu - \nu)(x_n, y_n)}{\mu x_n - T x_n, y_n} + (x_n, T^* y_n - \nu y_n) \leq \|\mu x_n - T x_n\| + \|T^* y_n - \nu y_n\| = \|\mu x_n - T x_n\| + \|T y_n - \nu y_n\| \to 0.$$

Thus, $(x_n, y_n) \to 0$, and we have a generalization of the following well-
known fact: for a normal operator, proper vectors belonging to distinct proper values are orthogonal. This suggests thinking of the sequences \( \{x_n\} \) and \( \{y_n\} \) as being “approximate proper vectors,” with their inner product defined to be \( \lim(x_n, y_n) \).

In what follows, we denote by \( \text{glim} \) a fixed “Banach generalized limit,” defined for bounded sequences \( \{\lambda_n\} \) of complex numbers (see page 34 of [1]); thus,

\[
\begin{align*}
(1) \quad \text{glim}(\lambda_n + \mu_n) &= \text{glim} \lambda_n + \text{glim} \mu_n, \\
(2) \quad \text{glim}(\lambda \lambda_n) &= \lambda \text{glim} \lambda_n, \\
(3) \quad \text{glim} \lambda_n &= \lim \lambda_n \text{ whenever } \{\lambda_n\} \text{ is convergent,} \\
(4) \quad \text{glim} \lambda_n &\geq 0 \text{ when } \lambda_n \geq 0 \text{ for all } n.
\end{align*}
\]

We shall not make use of a “translation-invariant” property of \( \text{glim} \); all we need are properties (1)–(4), in other words, a positive linear form on the vector space \( m \) of bounded sequences, which vanishes on the space \( c_0 \) of null sequences, and has the value 1 for the constant sequence \( \{1\} \). It follows from (1) and (4) that \( \text{glim} \lambda_n \) is real whenever \( \lambda_n \) is real for all \( n \); this implies in turn that \( \text{glim}(\lambda_n^*) = (\text{glim} \lambda_n)^* \) for any bounded sequence \( \{\lambda_n\} \).

3. An extension \( \mathcal{K} \) of \( \mathcal{K} \). Denote by \( \mathcal{B} \) the set of all sequences \( s = \{x_n\} \), with \( x_n \) in \( \mathcal{K} \) \( (n = 1, 2, 3, \ldots) \), such that \( \|x_n\| \) is bounded [that is, \( \{\|x_n\|\} \) is in \( m \)]. If \( s = \{x_n\} \) and \( t = \{y_n\} \), write \( s = t \) in case \( x_n = y_n \) for all \( n \). The set \( \mathcal{B} \) is a vector space relative to the definitions \( s + t = \{x_n + y_n\} \) and \( s\cdot t = \{s_n\cdot t_n\} \).

Suppose \( s = \{x_n\} \) and \( t = \{y_n\} \) belong to \( \mathcal{B} \); since \( \|(x_n, y_n)\| \leq \|x_n\| \|y_n\| \), it is permissible to define

\[
\phi(s, t) = \text{glim}(x_n, y_n).
\]

Evidently \( \phi \) is a positive symmetric bilinear functional on \( \mathcal{B} \) (see [2, §2]), hence \( |\phi(s, t)|^2 \leq \phi(s, s)\phi(t, t) \) (see [2, §5]). Let \( \mathcal{N} = \{s: \phi(s, s) = 0\} = \{s: \phi(s, t) = 0 \text{ for all } t \text{ in } \mathcal{B}\} \). Clearly \( \mathcal{N} \) is a linear subspace of \( \mathcal{B} \); we write \( s' \) for the coset \( s + \mathcal{N} \). The quotient vector space \( \mathcal{P} = \mathcal{B}/\mathcal{N} \) becomes an inner product space on defining \( (s', t') = \phi(s, t) \). Thus, if \( u = \{x_n\}' \) and \( v = \{y_n\}' \),

\[
(u, v) = \text{glim}(x_n, y_n).
\]

If \( x \) is in \( \mathcal{K} \), we write \( \{x\} \) for the sequence all of whose terms are \( x \), and \( x' \) for the coset \( \{x\}' + \mathcal{N} \). Evidently \( (x', y') = (x, y) \), and \( x \rightarrow x' \) is an isometric linear mapping of \( \mathcal{K} \) onto a closed linear subspace \( \mathcal{K}' \) of \( \mathcal{P} \). Regard \( \mathcal{P} \) as a linear subspace of its Hilbert space completion
4. A representation of $L(\mathcal{H})$. Every operator $T$ in $\mathcal{H}$ determines an operator $T^0$ in $\mathcal{K}$, as follows.

If $s = \{x_n\}$ is in $\mathcal{B}$, then the relation $\|Tx_n\|^2 \leq \|T\|^2 \|x_n\|^2$ shows that $\{Tx_n\}$ is in $\mathcal{B}$. Defining $T_0s = \{Tx_n\}$, we have a linear mapping $T_0: \mathcal{B} \to \mathcal{B}$ such that $\phi(T_0s, T_0s) \leq \|T\|^2 \phi(s, s)$. In particular, if $s$ is in $\mathcal{K}$, that is if $\phi(s, s) = 0$, then $T_0s$ is also in $\mathcal{K}$. It follows that $\{x_n\}' \to \{Tx_n\}'$ is a well-defined linear mapping of $\mathcal{B}$ into $\mathcal{B}$, which we denote $T^0$; thus, $T^0s' = (T_0s)'$, and the inequality $(T^0u, T^0w) \leq \|T\|^2(u, u)$, valid for all $u$ in $\mathcal{B}$, shows that $T^0$ is continuous, with $\|T^0\| \leq \|T\|$. Since in particular $T^0x' = (Tx)'$ for all $x$ in $\mathcal{H}$, it is clear that $\|T^0\| = \|T\|$, thus $\|T^0\| = \|T\|$. The continuous linear mapping $T^0$ extends to a unique operator in $\mathcal{K}$, which we also denote $T^0$.

The mapping $T \to T^0$ of $L(\mathcal{H})$ into $L(\mathcal{K})$ is easily seen to be a faithful *-representation: $(S + T)^0 = S^0 + T^0$, $(\lambda T)^0 = \lambda T^0$, $(ST)^0 = S^0T^0$, $(T^*)^0 = (T^0)^*$, $I^0 = I$, and $\|T^0\| = \|T\|$.

Suppose $T \geq 0$, that is, $(Tx, x) \geq 0$ for all $x$ in $\mathcal{K}$. If $u = \{x_n\}'$ is in $\mathcal{B}$, then $(Tx_n, x_n) \geq 0$ for all $n$, hence $(T^0u, u) = \text{glim}(Tx_n, x_n) \geq 0$; it follows that $(T^0v, v) \geq 0$ for all $v$ in $\mathcal{K}$. Clearly, for an operator $T$ in $\mathcal{K}$, one has $T \geq 0$ if and only if $T^0 \geq 0$.

**Lemma.** If $T$ is any operator in $\mathcal{K}$, $a(T^0) = a(T)$.

**Proof.** A complex number $\mu$ fails to belong to $a(T)$ if and only if there exists a number $\epsilon > 0$ such that $(T - \mu I)^* (T - \mu I) \geq \epsilon I$. By the above remarks, this condition is equivalent to $(T^0 - \mu I)^* (T^0 - \mu I) \geq \epsilon I$.

**Theorem 1.** For every operator $T$ in $\mathcal{K}$,

$$a(T) = a(T^0) = \rho(T^0).$$

**Proof.** The relations $a(T) = a(T^0) \supset \rho(T^0)$ have already been noted. Suppose $\mu$ is in $a(T)$. Choose a sequence $x_n$ in $\mathcal{K}$ such that $\|x_n\| = 1$ and $\|Tx_n - \mu x_n\| \to 0$, and set $u = \{x_n\}'$. Clearly $\|u\| = 1$ and $\|T^0u - \mu u\|^2 = \text{glim} \|Tx_n - \mu x_n\|^2 = 0$, hence $T^0u = \mu u$; that is, $\mu$ is in $\rho(T^0)$.

**Theorem 2.** If $T$ is any normal operator in $\mathcal{K}$, $T$ has an approximate proper value $\mu$ such that $|\mu| = \|T\|$.

**Proof.** Without loss of generality, we may suppose $\|T\| = 1$. If $1$ is in $s(T)$, the relation $s(T) = a(T)$ ends the proof. Let us assume henceforth that $I - T$ is invertible.
Let $S = T^*T$. Since $\|S\| = 1$, and since $S \succeq 0$, it follows from the remarks in \S 1 that 1 is an approximate proper value for $S$. By Theorem 1, 1 is a proper value for $S^0$. Let $\mathcal{M}$ be the null space of $S^0 - I$, thus $\mathcal{M} = \{v : S^0v = v \neq 0\}$. Since $TS = ST$ and $T^*S = ST^*$, $\mathcal{M}$ is invariant under $T^0$ and $(T^0)^*$; thus, $\mathcal{M}$ reduces $T^0$. We denote by $T^0/\mathcal{M}$ the restriction of $T^0$ to $\mathcal{M}$. Since $S^0/\mathcal{M} = I$, we have $(T^0/\mathcal{M})^*(T^0/\mathcal{M}) = (T^0S^0/\mathcal{M})(T^0/\mathcal{M}) = (T^0*T^0)/\mathcal{M} = S^0/\mathcal{M} = I$; clearly $T^0/\mathcal{M}$ is a unitary operator in $\mathcal{M}$. Write $U = T^0/\mathcal{M}$. Since $I - T$ has an inverse in $L(\mathcal{M})$, $I - T^0$ has an inverse in $L(\mathcal{M})$; since $\mathcal{M}$ reduces $I - T^0$, it follows that $I - U$ has an inverse in $L(\mathcal{M})$. Let $R$ be the Cayley transform of $U$, that is, $R = i(I + U)(I - U)^{-1}$; $R$ is a Hermitian operator in $\mathcal{M}$. Define $A = i(I + T)(I - T)^{-1}$; clearly $A^0/\mathcal{M} = R$.

Let $\alpha$ be any approximate proper value for $R$ (see \S 1). It is clear from the definition that $\alpha$ is also an approximate proper value for $A^0$. By Theorem 1, there is a nonzero vector $u$ in $\mathcal{M}$ such that $A^0u = \alpha u$. Since $A^0 = i(I + T^0)(I - T^0)^{-1}$, an elementary calculation gives $T^0u = (\alpha - i)(\alpha + i)^{-1}u$. Thus, $\mu = (\alpha - i)(\alpha + i)^{-1}$ belongs to $\rho(T^0) = \sigma(T^0) = a(T)$, and $|\mu| = 1 = \|T\|$. 

References


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