APPROXIMATE PROPER VECTORS

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1. **Notations and terminology.** Our terminology conforms with that of [2]. The inner product of vectors x and y in a Hilbert space $\mathcal{H}$ is denoted (x, y). An *operator* in $\mathcal{H}$ is a continuous linear mapping $T: \mathcal{H} \to \mathcal{H}$. The *-algebra of all operators in $\mathcal{H}$ is denoted $L(\mathcal{H})$. A complex number μ is a *proper value* for T if there exists a nonzero vector x such that $(T-\mu I)x=0$; such a vector x is a *proper vector* for T. A complex number μ is an *approximate proper value* for T in case there exists a sequence of vectors x_n such that $\|x_n\|=1$ and $\|Tx_n-\mu x_n\| \to 0$; equivalently, there does not exist a number $\epsilon>0$ such that $(T-\mu I)^*(T-\mu I) \geqq \epsilon I$.

The *spectrum* of an operator T, denoted $s(T)$, is the set of all complex numbers μ such that $T-\mu I$ has no inverse. The *approximate point spectrum* of T, denoted $a(T)$, is the set of all approximate proper values of T. The *point spectrum* of T, denoted $p(T)$, is the set of all proper values of T. Evidently $p(T) \subset a(T) \subset s(T)$. If T is normal, $s(T)=a(T)$ (see [2, Theorem 31.2]); if T is Hermitian, $a(T)$ contains a (necessarily real) number α such that $|\alpha| = \|T\|$ (see [2, Theorem 34.2]), and in particular one has an elementary proof of the fact that the spectrum of T is nonempty.

2. **Introduction.** The spectrum of a Hermitian operator is shown to be nonempty by completely elementary means. It would be nice to have an elementary proof for normal operators (see [2, p. 111]). The purpose of this note is to give a proof based on Banach limits. Incidentally, $\mathcal{H}$ will be extended to a curious Hilbert space $\mathcal{K}$, in which it becomes natural to speak of "approximate proper vectors."

Our motivation for the construction of $\mathcal{K}$ was as follows. Suppose T is a normal operator, and μ and ν are distinct approximate proper values of T. Choose sequences of unit vectors $\{x_n\}$ and $\{y_n\}$ such that $\|Tx_n-\mu x_n\| \to 0$ and $\|Ty_n-\nu y_n\| \to 0$. Then,

$$\begin{aligned} |(\mu-\nu)(x_n, y_n)| &= |(\mu x_n - Tx_n, y_n) + (x_n, T^*y_n - \nu^* y_n)| \\ &\leqq \|\mu x_n - Tx_n\| + \|T^* y_n - \nu^* y_n\| \\ &= \|\mu x_n - Tx_n\| + \|Ty_n - \nu y_n\| \to 0. \end{aligned}$$

Thus, $(x_n, y_n) \to 0$, and we have a generalization of the following well-

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known fact: for a normal operator, proper vectors belonging to distinct proper values are orthogonal. This suggests thinking of the sequences $\{x_n\}$ and $\{y_n\}$ as being "approximate proper vectors," with their inner product defined to be $\lim(x_n, y_n)$.

In what follows, we denote by glim a fixed "Banach generalized limit," defined for bounded sequences $\{\lambda_n\}$ of complex numbers (see page 34 of [1]); thus,

$$\text{glim}(\lambda_n + \mu_n) = \text{glim}\,\lambda_n + \text{glim}\,\mu_n, \tag{1}$$

$$\text{glim}(\lambda\lambda_n) = \lambda\,\text{glim}\,\lambda_n, \tag{2}$$

$$\text{glim}\,\lambda_n = \lim \lambda_n \text{ whenever } \{\lambda_n\} \text{ is convergent}, \tag{3}$$

$$\text{glim}\,\lambda_n \geqq 0 \text{ when } \lambda_n \geqq 0 \text{ for all } n. \tag{4}$$

We shall not make use of a "translation-invariant" property of glim; all we need are properties (1)–(4), in other words, a positive linear form on the vector space $\boldsymbol{m}$ of bounded sequences, which vanishes on the space $\mathbf{c}_0$ of null sequences, and has the value 1 for the constant sequence $\{1\}$. It follows from (1) and (4) that glim λ_n is real whenever λ_n is real for all n; this implies in turn that $\text{glim}(\lambda_n^*) = (\text{glim}\,\lambda_n)^*$ for any bounded sequence $\{\lambda_n\}$.

3. **An extension $\mathcal{K}$ of $\mathcal{H}$.** Denote by $\mathcal{B}$ the set of all sequences $s = \{x_n\}$, with x_n in $\mathcal{H}$ ($n = 1, 2, 3, \cdots$), such that $\|x_n\|$ is bounded [that is, $\{\|x_n\|\}$ is in $\boldsymbol{m}$]. If $s = \{x_n\}$ and $t = \{y_n\}$, write $s = t$ in case $x_n = y_n$ for all n. The set $\mathcal{B}$ is a vector space relative to the definitions $s + t = \{x_n + y_n\}$ and $\lambda s = \{\lambda x_n\}$.

Suppose $s = \{x_n\}$ and $t = \{y_n\}$ belong to $\mathcal{B}$; since $|(x_n, y_n)| \leqq \|x_n\|\,\|y_n\|$, it is permissible to define

$$\phi(s, t) = \text{glim}(x_n, y_n).$$

Evidently ϕ is a positive symmetric bilinear functional on $\mathcal{B}$ (see [2, §2]), hence $|\phi(s, t)|^2 \leqq \phi(s, s)\phi(t, t)$ (see [2, §5]). Let $\mathcal{N} = \{s : \phi(s, s) = 0\} = \{s : \phi(s, t) = 0 \text{ for all } t \text{ in } \mathcal{B}\}$. Clearly $\mathcal{N}$ is a linear subspace of $\mathcal{B}$; we write s' for the coset $s + \mathcal{N}$. The quotient vector space $\mathcal{P} = \mathcal{B}/\mathcal{N}$ becomes an inner product space on defining $(s', t') = \phi(s, t)$. Thus, if $u = \{x_n\}'$ and $v = \{y_n\}'$,

$$(u, v) = \text{glim}\,(x_n, y_n).$$

If x is in $\mathcal{H}$, we write $\{x\}$ for the sequence all of whose terms are x, and x' for the coset $\{x\} + \mathcal{N}$. Evidently $(x', y') = (x, y)$, and $x \rightarrow x'$ is an isometric linear mapping of $\mathcal{H}$ onto a closed linear subspace $\mathcal{H}'$ of $\mathcal{P}$. Regard $\mathcal{P}$ as a linear subspace of its Hilbert space completion

$\mathcal{K}$. Thus, $\mathcal{H}' \subset \mathcal{P} \subset \mathcal{K}$, where $\mathcal{H}'$ is a closed linear subspace of $\mathcal{K}$, and $\mathcal{P}$ is a dense linear subspace of $\mathcal{K}$.

4. **A representation of** $L(\mathcal{H})$. Every operator T in $\mathcal{H}$ determines an operator T^0 in $\mathcal{K}$, as follows.

If $s = \{x_n\}$ is in $\mathcal{B}$, then the relation $\|Tx_n\|^2 \leqq \|T\|^2 \|x_n\|^2$ shows that $\{Tx_n\}$ is in $\mathcal{B}$. Defining $T_0 s = \{Tx_n\}$, we have a linear mapping $T_0: \mathcal{B} \rightarrow \mathcal{B}$ such that $\phi(T_0 s, T_0 s) \leqq \|T\|^2 \phi(s, s)$. In particular if s is in $\mathcal{N}$, that is if $\phi(s, s) = 0$, then $T_0 s$ is also in $\mathcal{N}$. It follows that $\{x_n\}' \rightarrow \{Tx_n\}'$ is a well-defined linear mapping of $\mathcal{P}$ into $\mathcal{P}$, which we denote T^0; thus, $T^0 s' = (T_0 s)'$, and the inequality $(T^0 u, T^0 u) \leqq \|T\|^2 (u, u)$, valid for all u in $\mathcal{P}$, shows that T^0 is continuous, with $\|T^0\| \leqq \|T\|$. Since in particular $T^0 x' = (Tx)'$ for all x in $\mathcal{H}$, it is clear that $\|T^0\| \geqq \|T\|$, thus $\|T^0\| = \|T\|$. The continuous linear mapping T^0 extends to a unique operator in $\mathcal{K}$, which we also denote T^0.

The mapping $T \rightarrow T^0$ of $L(\mathcal{H})$ into $L(\mathcal{K})$ is easily seen to be a faithful *-representation: $(S+T)^0 = S^0 + T^0$, $(\lambda T)^0 = \lambda T^0$, $(ST)^0 = S^0 T^0$, $(T^*)^0 = (T^0)^*$, $I^0 = I$, and $\|T^0\| = \|T\|$.

Suppose $T \geqq 0$, that is, $(Tx, x) \geqq 0$ for all x in $\mathcal{H}$. If $u = \{x_n\}'$ is in $\mathcal{P}$, then $(Tx_n, x_n) \geqq 0$ for all n, hence $(T^0 u, u) = \text{glim}(Tx_n, x_n) \geqq 0$; it follows that $(T^0 v, v) \geqq 0$ for all v in $\mathcal{K}$. Clearly: for an operator T in $\mathcal{H}$, one has $T \geqq 0$ if and only if $T^0 \geqq 0$.

LEMMA. *If T is any operator in $\mathcal{H}$, $a(T^0) = a(T)$.*

PROOF. A complex number μ fails to belong to $a(T)$ if and only if there exists a number $\epsilon > 0$ such that $(T - \mu I)^*(T - \mu I) \geqq \epsilon I$. By the above remarks, this condition is equivalent to $(T^0 - \mu I)^*(T^0 - \mu I) \geqq \epsilon I$.

THEOREM 1. *For every operator T in $\mathcal{H}$,*

$$a(T) = a(T^0) = p(T^0).$$

PROOF. The relations $a(T) = a(T^0) \supset p(T^0)$ have already been noted. Suppose μ is in $a(T)$. Choose a sequence x_n in $\mathcal{H}$ such that $\|x_n\| = 1$ and $\|Tx_n - \mu x_n\| \rightarrow 0$, and set $u = \{x_n\}'$. Clearly $\|u\| = 1$ and $\|T^0 u - \mu u\|^2 = \text{glim} \|Tx_n - \mu x_n\|^2 = 0$, hence $T^0 u = \mu u$; that is, μ is in $p(T^0)$.

THEOREM 2. *If T is any normal operator in $\mathcal{H}$, T has an approximate proper value μ such that $|\mu| = \|T\|$.*

PROOF. Without loss of generality, we may suppose $\|T\| = 1$. If 1 is in $s(T)$, the relation $s(T) = a(T)$ ends the proof. Let us assume henceforth that $I - T$ is invertible.

Let $S=T^*T$. Since $\|S\|=1$, and since $S\geqq 0$, it follows from the remarks in §1 that 1 is an approximate proper value for S. By Theorem 1, 1 is a proper value for S^0. Let $\mathfrak{M}$ be the null space of S^0-I, thus $\mathfrak{M}=\{v: S^0v=v\}\neq\{0\}$. Since $TS=ST$ and $T^*S=ST^*$, $\mathfrak{M}$ is invariant under T^0 and $(T^0)^*$; thus, $\mathfrak{M}$ reduces T^0. We denote by $T^0/\mathfrak{M}$ the restriction of T^0 to $\mathfrak{M}$. Since $S^0/\mathfrak{M}=I$, we have $(T^0/\mathfrak{M})^*(T^0/\mathfrak{M})=(T^{0*}/\mathfrak{M})(T^0/\mathfrak{M})=(T^{0*}T^0)/\mathfrak{M}=S^0/\mathfrak{M}=I$; clearly $T^0/\mathfrak{M}$ is a unitary operator in $\mathfrak{M}$. Write $U=T^0/\mathfrak{M}$. Since $I-T$ has an inverse in $L(\mathfrak{H})$, $I-T^0$ has an inverse in $L(\mathfrak{K})$; since $\mathfrak{M}$ reduces $I-T^0$, it follows that $I-U$ has an inverse in $L(\mathfrak{M})$. Let R be the Cayley transform of U, that is, $R=i(I+U)(I-U)^{-1}$; R is a Hermitian operator in $\mathfrak{M}$. Define $A=i(I+T)(I-T)^{-1}$; clearly $A^0/\mathfrak{M}=R$.

Let α be any approximate proper value for R (see §1). It is clear from the definition that α is also an approximate proper value for A^0. By Theorem 1, there is a nonzero vector u in $\mathcal{P}$ such that $A^0u=\alpha u$. Since $A^0=i(I+T^0)(I-T^0)^{-1}$, an elementary calculation gives $T^0u=(\alpha-i)(\alpha+i)^{-1}u$. Thus, $\mu=(\alpha-i)(\alpha+i)^{-1}$ belongs to $p(T^0)=a(T^0)=a(T)$, and $|\mu|=1=\|T\|$.

References

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