A PROPERTY OF HOMOGENEOUS PROCESSES

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1. Statement of results. In the following $G$ is a locally compact Hausdorff group, $K$ a compact subgroup, $X = G/K$ the homogeneous space of left cosets, and $C(X)$ the Banach space of continuous complex valued functions on $X$ which are constant at infinity. $(P_t)_{t \geq 0}$ denotes a homogeneous process on $X$. That is, $P_t: C(X) \to C(X)$ is a strongly continuous one parameter semi-group of positive, constant-preserving linear transformations of $C(X)$ which commute with left translation by elements of $G$. Stated in other words, $(P_t)_{t \geq 0}$ is a strongly continuous one parameter semi-group on $C(X)$; $f \geq 0$ implies $P_tf \geq 0$; $P_t1 = 1$ and $L_gP_t = P_tL_g$ where $L_gf(x) = f(g^{-1}[x])$, $f \in C(X)$, $g \in G$, $x \in X$.

$P_t$ is represented by a kernel $P_t(x, A)$,

$$P_t f(x) = \int P_t(x, ds)f(s),$$

which is the transition probability of a stationary Markov process on $X$. For the kernel, homogeneity means $P_t(x, A) = P_t(g[x], g[A])$ when $x \in X$, $g \in G$ and $A$ is a Borel subset of $X$. It is shown below that

**Theorem.** Every homogeneous process possesses Property II.

**Property II.** For each $z \in X$ there is a regular Borel measure $Q_z$ on $X - \{z\}$ such that

$$t^{-1}P_t f(z) \to Q_z(f) \text{ as } t \to 0,$$

for each $f \in C(X)$ which vanishes on a neighborhood of $z$. $Q_z$ is not necessarily bounded but it is bounded on the complement of any neighborhood of $z$.

The stochastic and analytic implications of Property II are discussed in [1]. Roughly speaking, $Q_z$ describes very precisely the nature of the discontinuities in the paths of any process with transition probabilities $P_t(x, A)$, while from the analytic point of view $Q_z$ is related to the form of the infinitesimal generator of $P_t$, and Property II implies, for example, that the domain of this infinitesimal generator admits very satisfying smoothing operations.

2. Reduction and reformulation. By way of preliminary computations, let $H$ be a compact subgroup of $G$ and $dh$ the normalized Haar

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measure of $H$. Associated with $H$ there are two projection operators on $C(G)$. Namely, $f \mapsto R_H f$ and $f \mapsto L_H f$. These are defined by

\[
R_H f \cdot (g) = \int R_H f \cdot (g) \, dh, \quad R_H f \cdot (g) = f(gh).
\]

\[
L_H f \cdot (g) = \int L_H f \cdot (g) \, dh, \quad L_H f \cdot (g) = f(h^{-1}g).
\]

When $H$ is an invariant subgroup the automorphism $h \mapsto ghg^{-1}$, $g \in G$, preserves the normalized Haar measure, so that $\int f(gh) \, dh = \int f(hg) \, dh$ and $L_H = R_H$. Similar computations show that when $H$ is invariant $R_HK = L_HK$. A function $f \in C(G)$ is constant on the left cosets of $G$ modulo $K$ if and only if $R_K f = f$. Thus the two spaces $R_K C(G)$ and $C(X)$ are isomorphic, and using this isomorphism there is a one-to-one correspondence between homogeneous processes on $X$ and positive, constant-preserving semi-groups $(P_t)_{t \geq 0}$ on $C(G)$ which satisfy $L_g P_t = P_t L_g$, $g \in G$; $R_K P_t = P_t R_K = P_t$; and which are strongly continuous on $C(G)$ when $t > 0$ and on the subspace $R_K C(G)$ at $t = 0$. We call the latter a $K$-homogeneous process on $G$.

Because of the homogeneity it suffices to prove (1.1) for a fixed $x \in X$, say for the coset $K$ of $G/K$. Furthermore, it suffices to prove the limit (1.1) exists and is finite. The positivity of $P_t$ can then be used to show this limit has the form $Q_x(f)$ described in the statement of Property II. With these modifications the theorem can be restated.

**Restatement of the theorem.** If $P_t$ is a $K$-homogeneous process on $G$ and $f \in R_K C(G)$ vanishes on a neighborhood of $K$, then

\[
t^{-1} P_t f \cdot (e) \to \text{a finite limit as } t \to 0.
\]

One further reduction is necessary before proceeding. This is to notice that it suffices to prove the restatement of the theorem when $G$ is $\sigma$-compact (a countable union of compact sets). To see this let $A_t$ be a $\sigma$-compact set in $G$ on which the measure $P_t(e, \cdot)$ is concentrated, and let $G'$ be the subgroup of $G$ generated by some compact neighborhood $D$ of $K$ and the $A_t$, $t$-rational. $G'$ is closed and $\sigma$-compact and contains the support of every $P_t(e, \cdot)$, $t \geq 0$, because $P_{t} f \cdot (e)$ is continuous in $t > 0$. By homogeneity the support of the functional $f \mapsto P_{t} f \cdot (g)$, $g \in G'$, is also contained in $G'$ and, in fact, $P_{t}(g, A)$, $g \in G'$, $A$ Borel in $G'$, defines a $K$-homogeneous process on $G'$ with a unique extension to $G$. Clearly the limit (2.1) is unaffected by the values of $f$ outside $G'$ and one may as well assume $G$ is $\sigma$-compact.
3. **Proof of the theorem.** This theorem has already been proved when $X$ is separable in [1, §3]; so it is sufficient here to reduce the proof to the separable case. The key element in this proof is the following lemma which was suggested to the author by an argument in [2, p. 58].

**Lemma.** Let $G$ be a locally compact, Hausdorff, $\sigma$-compact topological group and let $f \in C(G)$. Then there is a compact invariant subgroup $N$ of $G$ such that $L_N f = f$ and $G/N$ is separable.

To prove (2.1) from the lemma, simply note that $P_t' = P_t L_N = L_N P_t$ defines an $NK$-homogeneous process on $G$ and that $f = L_N f = L_N R_{NK} f = R_{NK} f$ vanishes on a neighborhood of $NK$. Since $G/NK$ is separable, the separable version of (2.1) as proved in [1] implies that

$$t^{-1} P_t L_N f \cdot (e) = t^{-1} P_t f \cdot (e) \rightarrow \text{a limit} \quad \text{as } t \rightarrow 0.$$  

4. **Proof of the lemma.** $f$ is constant at infinity and hence uniformly continuous on $G$. Let $W_n$ be a compact neighborhood of the identity $e$ such that for every $g \in G$, $|f(gk) - f(g)| < 1/n$ when $k \in W_n$. We shall prove there is a compact invariant subgroup $N \subseteq \bigcap_n W_n$ such that $G/N$ is separable. Clearly for any such $N$, $L_N f = f$. To show the existence of $N$ let $C_n$ be an increasing sequence of compact sets which cover $G$ and choose $V_n$ inductively so that

1. $V_n$ is a compact symmetric neighborhood of $e$.
2. $V_n^2 \subseteq V_{n-1} \cap W_n$.
3. $g^{-1} V_n g \subseteq V_{n-1}$ for every $g \in C_n$.

$N = \bigcap_n V_n$ is a compact invariant subgroup of $G$. If $T$ is the canonical projection $G \rightarrow G/N$, the sets $T(V_n)$ form a basis for the neighborhoods at the identity in $G/N$ because for each open $U \supseteq N$, $V_n N - U$ is a decreasing sequence of compact sets with empty intersection and consequently $V_n N \subseteq U$ for some $n$. Since $G/N$ is a $\sigma$-compact uniform space with a countable basis for its uniformity it follows that $G/N$ is separable. Alternatively, $G/N$ is a $\sigma$-compact metrizable space and hence separable.

**References**


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