ON THE SIMULTANEOUS SOLUTION OF A CERTAIN SYSTEM OF LINEAR INEQUALITIES

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The author recently proved the following theorem:

THEOREM 1. *Let $\mathfrak{X}$ be a Hilbert space, with real or complex scalars and inner product $\langle x, y\rangle$. Let $x_1, \cdots, x_m$ and $y_1, \cdots, y_m$ be given such that*

$$\mathrm{Re}\langle x_i - x_j, y_i - y_j\rangle \geqq 0 \qquad (i, j = 1, \cdots, m), \tag{1}$$

and let x be any point of $\mathfrak{X}$. Then there exists a point y such that

$$\mathrm{Re}\langle x_i - x, y_i - y\rangle \geqq 0 \qquad (i = 1, \cdots, m). \tag{2}$$

The proof was patterned after Schoenberg's [3] proof of Kirszbraun's theorem. B. Grünbaum [2] has generalized my and Schoenberg's proofs to obtain a theorem which incorporates Theorem 1 and Kirszbraun's theorem, and J. G. Wendel has contributed a neater proof of Theorem 1. With Professor Wendel's permission, I reproduce his proof:

LEMMA. *Let $\mathfrak{X}$ be E^n, with the usual (real) scalars and inner product, and let $x_1, \cdots, x_m; y_1, \cdots, y_m$ be given such that*

$$\langle x_i - x_j, y_i - y_j\rangle \geqq 0 \qquad (i, j = 1, \cdots, m). \tag{1'}$$

Then there exists a point y such that

$$\langle x_i, y\rangle \leqq \langle x_i, y_i\rangle \qquad (i = 1, \cdots, m). \tag{2'}$$

PROOF OF THE LEMMA. Let A be the matrix whose ith row is x_i, and let b be the column-vector whose ith element is $\langle x_i, y_i\rangle$. Then (2′) is equivalent to $Ay \leqq b$. If there is no solution for y, then by Stiemke's Theorem [1, Theorem 2.7] there exists a row-vector $\eta \leqq 0$ such that

$$\eta A = 0 \quad \text{and} \quad \eta b = 1. \tag{3a, 3b}$$

Suppose this to be the case. Then (3a) implies $\sum_i \eta_i x_i = 0$, hence for each j,

$$\sum_i \eta_i \langle x_i, y_j\rangle = 0. \tag{4}$$

Also (3b) implies

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$$\sum_i \eta_i \langle x_i, y_i \rangle = 1. \tag{5}$$

Expanding (1′), multiplying by $\eta_i\eta_j$ and summing on i, j, we have

$$\begin{aligned} &\sum_i \eta_i \langle x_i, y_i \rangle \sum_j \eta_j + \sum_i \eta_i \sum_j \eta_j \langle x_j, y_j \rangle \\ &\qquad \geqq \sum_j \eta_j \sum_i \eta_i \langle x_i, y_j \rangle + \sum_i \eta_i \sum_j \eta_j \langle x_j, y_i \rangle \end{aligned} \tag{6}$$

i.e., by (4) and (5), $2 \sum_j \eta_j \geqq 0$. But $\eta \leqq 0$ and some $\eta < 0$ by (3b). We have a contradiction, and there is at least one solution for y.

PROOF OF THEOREM 1. First, we take up the case where $\mathfrak{X} = E^n$, and set $x_i' = x_i - x$; the conclusion follows by application of the Lemma to $x_1', \cdots, x_m'$ and $y_1, \cdots, y_m$.

We next suppose that $\mathfrak{X}$ is any finite-dimensional Hilbert space. If the scalars are complex, it is easily verified that $[x, y] = \text{Re}\langle x, y \rangle$ is *real* inner product provided the scalar product $\alpha \cdot x$ is restricted to real α, and that the resulting Hilbert space with real scalars is of dimension $2n$. The conclusion now follows from the isomorphism with E^n or E^{2n}.

In case $\mathfrak{X}$ is infinite-dimensional, we simply apply the above results to the (finite-dimensional) subspace spanned by $x_1, \cdots, x_m$; $y_1, \cdots, y_m$ and x.

REFERENCES

1. D. Gale, *The theory of linear economic models*, McGraw-Hill, New York, 1960.

2. B. Grünbaum, *A generalization of theorems of Kirszbraun and Minty*, Proc. Amer. Math. Soc. (to appear).

3. I. J. Schoenberg, *On a theorem of Kirzbraun and Valentine*, Amer. Math. Monthly **60** (1953), 620–622.

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