A NOTE ON LOGARITHMS OF NORMAL OPERATORS[1]

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All operators considered in this note are bounded and defined on a fixed Hilbert space X. In [**4**], C. R. Putnam has proved that if H is a positive definite selfadjoint operator and $\exp T=H$, then $\|T\| \leqq 2 \ln 2$ implies that T is a selfadjoint operator. In Theorem 3 we prove that it is sufficient to assume that $\|T\|<2\pi$ in order that T be selfadjoint. This condition, already in the set of complex numbers, cannot be replaced by $\|T\| \leqq 2\pi$ without changing the conclusion. In Theorem 4 we prove that $\|T\| \leqq 2\pi$ in a finite dimensional space implies that T is a normal operator. In Theorem 2 some conditions for a logarithm T of a normal operator N are derived. Assuming that the spectrum of N lies in the set

$$\Omega = \left\{ re^{i\phi} \mid -\alpha\pi \leqq \phi \leqq \alpha\pi,\ 0 \leqq \alpha \leqq \frac{1}{2},\ r \geqq \epsilon > 0 \right\} \tag{1}$$

we prove that $\exp T=N$ and $\|T\|<(1-\alpha^2/4)\pi$ imply that T is a normal operator. All these results are consequences of

THEOREM 1. *Let X be a Hilbert space and $T: X \to X$ a bounded operator such that* $\exp T=N$ *is a normal operator. Then*

$$T = N_0 + 2\pi i W, \tag{2}$$

with

$$N_0 = \int \ln \lambda dE(\lambda), \tag{3}$$

where $\ln \lambda$ *is the principal (or any) branch of the logarithmic function and $E(\lambda)$ is the spectral measure of N. The bounded operator W commutes with N_0 and there exists a bounded and regular, positive definite selfadjoint operator Q such that*

$$W_0 = Q^{-1} W Q \tag{4}$$

is a selfadjoint operator the spectrum of which belongs to the set of all integers (cf. [**2**, *Theorem* **4**; **3**, *Lemma* 2.2 *and Theorem* III]).

PROOF. The operator N_0 defined by (3) is bounded and $\exp N_0=N$.

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Since T commutes with N, it commutes with $E(\lambda)$ [**1**, p. 68, Theorem 2] and therefore T commutes also with N_0. Thus the operator $2\pi i W = T - N_0$ has the property that $\exp 2\pi i W = E$, where E is the identity operator. Now, the one-parameter group

$$G(t) = \exp 2\pi i t W \tag{5}$$

with $t \in R$ (R denotes the set of all real numbers), is continuous and periodic. Thus

$$\sup_{t \in R} \|G(t)\| < +\infty \tag{6}$$

which, by the well-known theorem of B. Sz. Nagy [**5**], implies the existence of a regular, positive definite, selfadjoint operator Q such that

$$Q^{-1}G(t)Q = \exp 2\pi i t Q^{-1}WQ$$

is a group of unitary operators. But this is possible if and only if the operator

$$W_0 = Q^{-1}WQ$$

is selfadjoint. Now $\exp 2\pi i W_0 = E$ and the fact that W_0 is selfadjoint imply that the spectrum of W_0 consists of integers only. Q.E.D.

THEOREM 2. *Let X be a Hilbert space, $N: X \to X$ a bounded normal operator the spectrum of which is contained in the set* (1). *If $T: X \to X$ is a bounded operator such that*

$$\exp T = N \quad \text{and} \quad \|T\| < \left(1 - \frac{\alpha^2}{4}\right)\pi,$$

then T is a normal operator.

PROOF. Let $E(\lambda)$ be the spectral measure of N and

$$N_0 = \int_\Omega \ln \lambda dE(\lambda)$$

where $\ln \lambda = \ln |\lambda| + i \arg \lambda$, $-\pi \leqq \arg \lambda < \pi$. According to Theorem 1 we have $T = N_0 + 2\pi i W$. We will prove that $W = 0$. First of all we have:

$$\max(\|N\|, \|N^{-1}\|) < \exp\left(1 - \frac{\alpha^2}{4}\right)\pi.$$

This, the definition of N_0, and

$$\|N_0\| = \sup_{\lambda\in\sigma(N)} |\ln\lambda| < \left[\alpha^2\pi^2 + \left(1 - \frac{\alpha^2}{4}\right)^2\pi^2\right]^{1/2},$$

where $\sigma(N)$ denotes the spectrum of N [1, p. 62, Theorem 2], imply:

$$2\pi\|W\| \leqq \|T\| + \|N_0\| < \pi\left(1 - \frac{\alpha^2}{4}\right) + \pi\left[\alpha^2 + \left(1 - \frac{\alpha^2}{4}\right)^2\right]^{1/2}$$
$$= 2\pi.$$

Thus

$$\|W\| < 1.$$

On the other hand $W_0 = Q^{-1}WQ$, where Q and W_0 are selfadjoint operators and the spectrum of W_0 is contained in the set of all integers. If $W_0x = nx$, then $Wy = ny$, with $y = Qx/\|Qx\|$. But then $\|W\| \geqq |n|$ which by virtue of (7) gives $n = 0$. Thus the spectrum of W_0 consists of the origin only. This implies $W_0 = 0$ and therefore $W = 0$ which implies $T = N_0$. Q.E.D.

REMARK 1. In a two dimensional unitary space the operator T, which in an orthonormal basic set has the matrix

$$i\begin{pmatrix} 1 & 0 \\ 1 & 1 - 2\pi \end{pmatrix},$$

possesses the following properties: (a) T is not normal, (b) $\exp T = Ee^i$ and (c) $\|T\| < 2\pi$. This example shows that in general the normality of $N = \exp T$ and $\|T\| < 2\pi$ does not imply that T is a normal operator. But in the case in which N is a positive definite, selfadjoint operator this is true, because we have:

THEOREM 3. *Let X be a Hilbert space, $H: X \to X$ a bounded, positive definite, selfadjoint operator and $T: X \to X$ an operator such that*

$$\exp T = H \quad \textit{and} \quad \|T\| < 2\pi.$$

Then the operator T is a selfadjoint operator.

PROOF. The operator $H_0 = N_0$ defined by (3), with $\ln\lambda$ as the principal branch of the logarithm, is a bounded selfadjoint operator and $T = H_0 + 2\pi iW$. Again, if $x \neq 0$, $W_0x = Q^{-1}WQx = nx$, then $Wy = ny$, with $y = Qx/\|Qx\|$, implies

$$(Ty, y) = (H_0y, y) + 2\pi in.$$

Since H_0 is selfadjoint, (H_0y, y) is real and we get:

$$|(Ty, y)|^2 = (H_0y, y)^2 + 4\pi^2n^2 \leqq \|T\|^2 < 4\pi^2.$$

Thus $n^2<1$, which implies $n=0$. This obviously leads to $W=0$ and therefore $T=H_0$. Q.E.D.

COROLLARY 1. *If T is a bounded linear operator and H a positive definite, selfadjoint operator such that*

$$\exp T = He^{i\theta},$$

with $\theta\in[0, 2\pi]$, then

$$\|T\| \geqq \theta \text{ if } \theta \in [0, \pi] \quad \text{and} \quad \|T\| \geqq \pi - \theta \text{ if } \theta \in [\pi, 2\pi].$$

PROOF. Suppose that $\theta\in[0, \pi]$ and $\|T\|<\theta$. Then $\|T-i\theta E\|<2\pi$ and $\exp(T-i\theta E)=H$, together with Theorem 3, imply that $H_0=T-i\theta E$ is a selfadjoint operator. But then $|(Tx, x)|^2=(H_0x, x)^2+\theta^2 \leqq\|T\|^2<\theta^2$ holds for any $x\in X$, $\|x\|=1$. Since this is impossible, we have $\|T\|\geqq\theta$. If $\theta\in[\theta, 2\pi]$, then $2\pi-\theta\in[0, \pi]$ and we have $\|T\| =\|T^*\|\geqq 2\pi-\theta$ because of $\exp T^*=He^{i(2\pi-\theta)}$.

THEOREM 4. *Let X be a finite dimensional unitary space, $H: X\rightarrow X$ a positive definite, selfadjoint operator and $T: X\rightarrow X$ an operator such that*

$$\exp T = H \quad \text{and} \quad \|T\| \leqq 2\pi. \tag{8}$$

Then T is a normal operator.

PROOF. Since H is selfadjoint, we have

$$H = \sum_{k=1}^{m} \lambda_k E_k,$$

where λ_k are positive numbers, $\lambda_k\neq\lambda_{k'}$ if $k\neq k'$, and E_k are selfadjoint pairwise orthogonal projections. If X_k denotes the range of E_k, then X is an orthogonal sum of the subspaces X_k. Since T commutes with H, we have

$$T = \sum_{k=1}^{m} \oplus T_k,$$

where T_k is the restriction of T to X_k and the symbol $\oplus$ denotes the orthogonal sum of operators. Now (8) becomes:

$$\exp T_k = \lambda_k E_k, \qquad \|T_k\| \leqq 2\pi, \qquad k = 1, 2, \cdots, m. \tag{9}$$

If we prove that every T_k is normal, then T as the orthogonal sum of normal operators will be normal.

Therefore it is sufficient to prove, that if X is an n-dimensional unitary space, $\exp T=\lambda E$, $\lambda>0$ and $\|T\|\leqq 2\pi$, then T is normal. According to Theorem 1 we have $T=E \ln \lambda+2\pi iW$ with $\ln \lambda$ real for

$\lambda > 0$. Now, $Wy = ny$, $\|y\| = 1$, implies $Ty = \ln y + 2\pi i n y$ and therefore

$$\|Ty\|^2 = (\ln \lambda)^2 + 4\pi^2 n^2 \leqq 4\pi^2.$$

If $\ln \lambda \neq 0$, then $n = 0$ and $T = E \ln \lambda$, i.e. T is selfadjoint. On the other hand, if $\ln \lambda = 0$, then $T = 2\pi i W$, the spectrum of W is contained in the set $\{-1, 0, 1\}$ and $\|W\| \leqq 1$. To prove that W is normal we take an orthonormal basic set $e_1, \cdots, e_n$ in X such that the matrix $W(e)$ which belongs to W in this basic set has the property that $[W(e)]_{ij} = 0$ if $1 \leqq i < j \leqq n$. Now

$$\|We_1\|^2 = |[W(e)]_{11}|^2 + |[W(e)]_{21}|^2 + \cdots + |[W(e)]_{n1}|^2 \leqq 1.$$

Hence, if $|[W(e)]_{11}| = 1$, then $[W(e)]_{21} = \cdots = [W(e)]_{n1} = 0$. This method implies that the matrix $W(e)$ has the following form:

$$W(e) = \begin{pmatrix} W_0(e) & 0 \\ 0 & W_1(e) \end{pmatrix},$$

where $W_0(e)$ is a diagonal matrix and all elements of the matrix $W_1(e)$ on and above the main diagonal are zero. Obviously $\exp 2\pi i W_1(e) = E_1(e)$, where $E_1(e)$ is the unit matrix of the same order as $W_1(e)$. Since $W_1(e)$ has all eigenvalues zero, and since it is similar to a hermitian matrix (Theorem 1), we find $W_1(e) = 0$. Thus the matrix $W(e)$ is diagonal, i.e., the operator W is a normal operator. Q.E.D.

REMARK 2. In a two-dimensional unitary space, the operator T, which in an orthonormal basic set has the matrix

$$2\pi i \begin{pmatrix} 1 & 0 \\ \epsilon & 0 \end{pmatrix}, \qquad \epsilon \neq 0,$$

has the following properties: (1) T is not normal, (2) $\exp T = E$ and (3) $\|T\| = 2\pi(1+\epsilon^2)^{1/2}$, i.e., the condition $\|T\| \leqq 2\pi$ of Theorem 4 cannot be weakened without altering the conclusion.

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