A NOTE ON THE BOUNDEDNESS OF SOLUTIONS OF LINEAR PARABOLIC EQUATIONS

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Hartman and Wintner [1] obtained a Sturmian comparison theorem for self-adjoint second order elliptic equations of the form

\[ \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) + fu = 0, \quad a_{ij} = a_{ji}, \]

in a bounded domain \( B \) with boundary \( \partial B \). In this note, their method is slightly modified to prove the following theorem.

Denote by \( D \) the semi-infinite cylinder \( \{(x, t) : x \in B, t > 0\} \), by \( \bar{D} \) its closure and by \( D_T \) the intersection of \( D \) with the half-space \( t \leq T \). Suppose \( u = u(x) \) is a solution of equation (1) which is continuous in \( \bar{B} \) the closure of \( B \), vanishes on \( \partial B \) and has continuous second derivatives in \( B \). Again, suppose \( w = w(x, t) \) is defined and continuous in the closed region \( \bar{D}_T \), is positive on \( B \) at \( t = 0 \) and on \( \partial B \) for all \( t \geq 0 \) and has continuous derivatives \( \partial w/\partial t, \partial^2 w/\partial x_i \partial x_j \) which satisfy the parabolic equation

\[ \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( A_{ij} \frac{\partial w}{\partial x_j} \right) + Fw = C \frac{\partial w}{\partial t}, \quad A_{ij} = A_{ji}, \]

in \( D_T \) for all \( T > 0 \). The functions \( A_{ij}, \partial A_{ij}/\partial x_i, F \) and \( C \) are uniformly bounded continuous functions of \( x_i \) and \( t \) in \( D_T \) for any given \( T > 0 \), while \( C \) is bounded in \( D \) by two positive constants \( C_0 \) and \( C_1 \) \((0 < C_0 \leq C \leq C_1)\) and the quadratic form \( \sum_{i,j} A_{ij} \xi_i \xi_j \) is non-negative at all points in \( D \).

**Theorem 1.** If \( \int_B \left\{ u^2 (F - f) + \sum_{i,j=1}^{n} (a_{ij} - A_{ij}) (\partial u/\partial x_i) (\partial u/\partial x_j) \right\} \, dt \geq \epsilon(t) \) for all \( t > 0 \) and \( \int_0^T \epsilon(t) \, dt \) tends to infinity with \( T \), then \( w \) is unbounded in \( D \).

**Proof.** Since \( w > 0 \) on \( \bar{B} \) at \( t = 0 \) and on \( \partial B \) for all \( t > 0 \), the maximum principle for parabolic equations (see [2]) implies \( w > 0 \) in \( D_T \) for all \( T > 0 \).

The Green identity leads from the boundary condition \( u = 0 \) on \( \partial B \), to the divergence relation

\[ \int_B \left\{ \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (u^2 h) \right\} \, dt = 0 \quad \text{for all } t \geq 0, \]

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where \( h^i \) is the \( t \)-dependent vector function

\[
h^i = \sum_{j=1}^{n} A_{ij} \frac{\partial w}{\partial x_j}.
\]

This vector is finite at each point of \( D \) since \( w > 0 \) in this region.

Since \( u \) is a solution of (1) and vanishes on \( \partial B \), another application of Green's identity shows that

\[
\int_B \left\{ \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - f u^2 \right\} d\tau = 0.
\]

It is clear from (4) and (2) that

\[
\sum_{i=1}^{n} \frac{\partial h^i}{\partial x_i} = C \frac{\partial w}{\partial t} - F - \sum_{i,j=1}^{n} \frac{A_{ij}}{w^2} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j}
\]

and using this in conjunction with (3) and (5), we see that

\[
\int_B \frac{u^2 C}{w} \frac{\partial w}{\partial t} d\tau = \int_B \left\{ \sum_{i,j=1}^{n} \left( a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - 2 \frac{u}{w} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_j} \right. \right.
\]

\[
\left. + \frac{u^2}{w^2} A_{ij} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \right\} + u^2(F - f) \right\} d\tau,
\]

for all \( t \geq 0 \). The right-hand side of this equation can be rewritten in the form

\[
\int_B \left\{ \sum_{i,j=1}^{n} A_{ij} \left( \frac{\partial u}{\partial x_i} - \frac{u}{w} \frac{\partial w}{\partial x_i} \right) \left( \frac{\partial u}{\partial x_j} - \frac{u}{w} \frac{\partial w}{\partial x_j} \right) \right\} d\tau
\]

\[
+ \int_B \left\{ u^2(F - f) + \sum_{i,j=1}^{n} (a_{ij} - A_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right\} d\tau,
\]

since \( A_{ij} = A_{ji} \).

Since the quadratic form \( \sum_{i,j=1}^{n} A_{ij} \xi_i \xi_j \) is positive definite or at least non-negative and the second term of expression (8) is assumed greater than or equal to \( \epsilon(t) \), we see from (7) that \( \frac{\partial z}{\partial t} \geq \epsilon(t) \), where \( z = \int_B C u^2 \log w d\tau \). But then \( z(T) \geq z(0) + \int_0^T \epsilon(t) dt \) and is unbounded if \( \int_0^T \epsilon(t) dt \) tends to infinity with \( T \). If \( z \) is unbounded, \( w \) cannot be bounded; for if it were bounded by \( w_0 \) for all \( T \) we would have \( z(T) \leq \int_B C u^2 \log w d\tau \) always.

Protter [3] extended the comparison theorem of Hartman and Wintner to a form valid for a pair of general linear elliptic equations. A corresponding extension of Theorem 1 is given below.

**Theorem 2.** If \( u \) and \( w \) satisfy the general linear equations
\begin{align}
(9) \quad \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + f u = 0, \\
(10) \quad \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( A_{ij} \frac{\partial w}{\partial x_j} \right) + \sum_{i=1}^{n} B_i \frac{\partial w}{\partial x_i} + F w = C \frac{\partial w}{\partial t},
\end{align}

in \( B \) and \( D \), and the boundary and continuity conditions of Theorem 1, then \( w \) is unbounded in \( D \) if \( \int_0^T \varepsilon(t) \, dt \) tends to infinity with \( T \) and

\[
\int_B \left\{ \sum_{i,j=1}^{n} \left( a_{ij} - A_{ij} \right) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{u^2}{4} \left[ (F - f) - \sum_{i,j=1}^{n} A_{ij} B_{ij} + 2 \sum_{i=1}^{n} \left( \frac{\partial b_i}{\partial x_i} - \frac{\partial B_i}{\partial x_i} \right) \right] \right\} \, d\tau
\]

is greater than or equal to \( \varepsilon(t) \) for all \( t > 0 \). \( A_{ij} \) denote the elements of the inverse matrix of \( A_{ij} \) and the functions \( \partial b_i/\partial x_i \), \( \partial B_i/\partial x_i \) are assumed continuous in \( B \) and \( D \) respectively.

**Proof.** Let us construct an integral corresponding to (7) with its left-hand side, like (8), consisting of an integral of a non-negative form containing all terms involving the function \( w \), and a residual integral containing only terms involving the function \( u \) and the coefficients of equations (9) and (10).

Consider the expression

\[
E = \int_B \left\{ \sum_{i,j=1}^{n} \left( a_{ij} - A_{ij} \right) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{i=1}^{n} u \frac{\partial u}{\partial x_i} \left( B_i - b_i \right) + (F - f) u^2 \right\} \, d\tau
\]

\[
+ \int_B \left\{ \sum_{i,j=1}^{n} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \sum_{i=1}^{n} u \frac{\partial u}{\partial x_i} B_i - F u^2 \right\} \, d\tau.
\]

The term \( F u^2 \) of the second integral can be eliminated by subtracting from it the vanishing integral

\[
E' = \int_B \left\{ \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( u^2 \sum_{j=1}^{n} \frac{A_{ij}}{w} \frac{\partial w}{\partial x_j} \right) \right\} \, d\tau
\]

\[
= \int_B \left\{ \frac{C u^2}{w} \frac{\partial w}{\partial t} - F u^2 - \frac{u^2}{w} \sum_{i=1}^{n} B_i \frac{\partial w}{\partial x_i} - \sum_{i,j=1}^{n} \left( \frac{u^2}{w^2} A_{ij} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} - 2 \frac{u}{w} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_j} \right) \right\} \, d\tau.
\]
Since $E$ is zero by virtue of (9),

$$
\frac{\partial z}{\partial t} = \int_B \frac{Cu^2}{w} \frac{\partial w}{\partial t} \, d\tau
$$

$$
= \int_B \left\{ \sum_{i,j=1}^{n} (a_{ij} - A_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{i=1}^{n} u \frac{\partial u}{\partial x_i} (B_i - b_i) + (F - f)u^2 \right\} \, d\tau
$$

$$
+ \sum_{i=1}^{n} u \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_i}
$$

$$
+ \int_B \left\{ \sum_{i,j=1}^{n} A_{ij} \left( \frac{\partial u}{\partial x_i} - u \frac{\partial w}{\partial x_i} \right) \left( \frac{\partial u}{\partial x_j} - u \frac{\partial w}{\partial x_j} \right) \right\} \, d\tau
$$

$$
- \sum_{i=1}^{n} uB_i \left( \frac{\partial u}{\partial x_i} - u \frac{\partial w}{\partial x_i} \right) \, d\tau
$$

(14)

$$
= \int_B \left\{ \sum_{i,j=1}^{n} (a_{ij} - A_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{u^2}{4} \left[ A(F - f) - A^{ij}B_iB_j + 2 \sum_{i=1}^{n} \left( \frac{\partial b_i}{\partial x_i} - \frac{\partial B_i}{\partial x_i} \right) \right] \right\} \, d\tau
$$

$$
+ \int_B \left\{ \sum_{i,j=1}^{n} A_{ij} \left( \frac{\partial u}{\partial x_i} - u \frac{\partial w}{\partial x_i} - \sum_{k=1}^{n} A^{ik}B_k \right) \frac{\partial u}{\partial x_j} \frac{\partial w}{\partial x_j} \right\} \, d\tau
$$

where $A^{ij}$ is the inverse matrix of $A_{ij}$. The matrix $A_{ij}$ is required to be non-singular in Theorem 2 as well as non-negative in $D$. Since $\sum_{i,j=1}^{n} A_{ij} \xi_i \xi_j$ is positive definite in $D$ the second integral of (14) is non-negative. The remainder of the proof now proceeds as for Theorem 1.

**Bibliography**


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