

AN EXAMPLE OF ANOMALOUS SINGULAR HOMOLOGY

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Introduction. The Čech homology groups of an r -dimensional space are known to vanish in dimensions greater than r . We will show that the corresponding assertion for the singular homology groups is false, even on the category of locally $(r-1)$ -connected compacta. An r -dimensional compactum X , locally contractible save at one point, is given such that the singular homology groups $H_q(X; Q)$ with rational coefficients are nontrivial for infinitely many values of q . This settles a question raised by Eilenberg and Steenrod [1, Problem 22].

Let X denote the union of a countable number of r -spheres ($r > 1$) with a single point in common and a metric topology such that the diameter of the spheres tends to zero with increasing index. This example was suggested by Steenrod. Our result is

THEOREM 1. *The rational singular homology groups $H_q(X; Q)$ with $q \equiv 1 \pmod{r-1}$, $q > 1$, are not zero.² In fact these groups are not even countable.*

The proof, which applies to bouquets of a more general nature, is based on the composition

$$\omega_Q: \pi_q(X) \rightarrow H_q(X; Q)$$

of the Hurewicz homomorphism $\omega: \pi_q(X) \rightarrow H_q(X)$ and the coefficient homomorphism induced by the inclusion $Z \rightarrow Q$ of the integers in the rationals. It is shown that ω_Q is not trivial.

1. Two observations. If X is any simply-connected space, the kernel of the Hurewicz homomorphism $\omega: \pi_q(X) \rightarrow H_q(X)$, can be characterized as follows. Let α be an element of $\pi_q(X)$, where $q > 1$.

LEMMA 1. *$\omega(\alpha) = 0$ if and only if there exists a finite polyhedron K of dimension $< q$, and a map $f: K \rightarrow X$ such that $\alpha \in f_*\pi_q(K)$.*

PROOF. In the exact sequence

$$\dots \rightarrow \Gamma_q(SX) \rightarrow \pi_q(SX) \xrightarrow{\omega} H_q(SX) \rightarrow \dots$$

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² It seems plausible to conjecture that $H_q(X) = 0$ for $q \not\equiv 1 \pmod{r-1}$, $q > 0$. It is not difficult to prove this for $r < q < 2r - 1$.

(see [3]; here SX denotes the singular complex of X as there defined), the image of Γ_q in π_q is the image of $\pi_q((SX)^{q-1})$ in $\pi_q(SX)$ under the inclusion of the $(q-1)$ -skeleton $(SX)^{q-1}$ of SX in SX . Since the image of a map $S^q \rightarrow (SX)^{q-1}$ is compact, it lies in a finite subcomplex $K \subset (SX)^{q-1}$. Therefore the lemma is true for SX . It follows for X also, since the natural map $SX \rightarrow X$ induces isomorphisms of the homotopy and singular homology groups.

Let $k: S^q \rightarrow S^m \vee S^n$ ($q = m + n - 1$) represent the Whitehead product $[\iota_m, \iota_n]$ of the generators of $\pi_m(S^m), \pi_n(S^n)$; let a_m, b_n similarly generate $H^m(S^m; \Lambda), H^n(S^n; \Lambda)$, regarded as subgroups of the corresponding groups of $S^m \vee S^n$, where Λ is a ring. Then $a_m \cup b_n = 0$, and the functional cup-product [2]

$$c_q = a_m \cup_k b_n \in H^q(S^q; \Lambda)$$

is defined and generates this cohomology group.

More generally, if $\alpha \in \pi_m(X), \beta \in \pi_n(X)$, the Whitehead product $[\alpha, \beta]$ can be represented by a map $f = g \circ k$:

$$S^q \xrightarrow{k} S^m \vee S^n \xrightarrow{g} X.$$

Given cohomology classes $u \in H^m(X; \Lambda), v \in H^n(X; \Lambda)$ with $u \cup v = 0$, the functional cup-product

$$u \cup_f v \in H^q(S^q; \Lambda)$$

is defined and equal to $g^*u \cup_k g^*v$. This is because $f^*: H^q(X; \Lambda) \rightarrow H^q(S^q; \Lambda)$ is trivial, since k^* is necessarily trivial on $H^q(S^m \vee S^n; \Lambda)$.

LEMMA 2. $u \cup_f v = \lambda c_q$, where c_q generates $H^q(S^q; \Lambda)$ and

$$\lambda = \begin{cases} \langle u, \omega\alpha \rangle \langle v, \omega\beta \rangle & \text{if } n \neq m, \\ \langle u, \omega\alpha \rangle \langle v, \omega\beta \rangle + (-1)^n \langle u, \omega\beta \rangle \langle v, \omega\alpha \rangle & \text{if } n = m. \end{cases}$$

As usual, $\langle u, z \rangle$ denotes $h(u)(z)$, where $h: H^p(X; \Lambda) \rightarrow \text{Hom}(H_p(X); \Lambda)$ is the projection. The proof is easy.

2. **Bouquets with the strong topology.** Let $(A_\xi)(\xi \in \mathbb{Z})$ be a family of spaces, each with a specified base point $*$, indexed to an infinite set \mathbb{Z} . The bouquet

$$Y = \bigvee_{\xi \in \mathbb{Z}} A_\xi$$

will mean the subset of the cartesian product $\prod_{\xi \in \mathbb{Z}} A_\xi$ which consists of all points (a_ξ) such that $a_\xi \neq *$ for at most one value of ξ . We identify each A_ξ with the corresponding subset of Y . The result-

ing topology is stronger than the usual weak topology; this space has the properties:

(1) There are embeddings $A_\eta \rightarrow Y$ and projections $Y \rightarrow A_\zeta$, such that the compositions $A_\eta \rightarrow Y \rightarrow A_\zeta$ are the identity maps if $\eta = \zeta$ and trivial otherwise.

(2) Any neighbourhood of the common point $*$ contains all but a finite number of the A_ξ .

From (1) it follows that we may regard $\pi_q(A_\xi, *)$, $H_q(A_\xi)$, $H^q(A_\xi; Q)$ as embedded as direct summands in $\pi_q(Y, *)$, $H_q(Y)$, $H^q(Y; Q)$. From (2) it follows that if $J = \{1, 2, 3, \dots\} \subset \mathbb{Z}$ is a countable subset, and if elements $\alpha_n \in \pi_q(A_n, *)$, $n \in J$, are given, we may form the sum

$$\alpha = \sum_{n \in J} \alpha_n$$

(the order being unimportant if $q > 1$). This can be defined by a representative map $f: (I^q, \dot{I}^q) \rightarrow (Y, *)$ given by

$$f(t_1, \dots, t_q) = f_n(n(n+1)(t_1 - 1) + n + 1, t_2, \dots, t_q)$$

for $1 - 1/n \leq t_1 \leq 1 - 1/(n+1)$, $n = 1, 2, 3, \dots$, where f_n is a representative map for α_n .

Now let A_i, B_i ($i \in J = \{1, 2, 3, \dots\}$) be simply-connected spaces with base points; then

$$X = \bigvee_{i \in J} (A_i \vee B_i) = \left(\bigvee_{i \in J} A_i \right) \vee \left(\bigvee_{i \in J} B_i \right)$$

is also simply-connected. Let $m, n > 1$ be fixed integers, and as before let $q = m + n - 1$.

THEOREM 2. *Let $\alpha_i \in \pi_m(A_i, *)$, $\beta_i \in \pi_n(B_i, *)$, and $\gamma = \sum_{i \in J} [\alpha_i, \beta_i] \in \pi_q(X, *)$. Then $\omega_q(\gamma) \neq 0$ if $\omega_q(\alpha_i) \neq 0$, $\omega_q(\beta_i) \neq 0$ for an infinite number of values of i . (We recall that ω_q is the composition*

$$\pi_r(X, *) \rightarrow H_r(X) \rightarrow H_r(X; Q)\omega.$$

PROOF. If $\omega_q(\alpha_i) \in H_m(A_i; Q)$ is not zero, we may choose a cohomology class $u_i \in H^m(A_i; Q)$ such that $\langle u_i, \omega_q(\alpha_i) \rangle \neq 0$. Similarly if $\omega_q(\beta_i) \neq 0$ we can choose $v_i \in H^n(B_i; Q)$ such that $\langle v_i, \omega_q(\beta_i) \rangle \neq 0$.

First suppose that $\omega(\gamma) \in H_q(X)$ is zero: we derive a contradiction. By Lemma 1 there is a map $f: S^q \rightarrow X$ representing the homotopy class γ , and which can be factorised through a map g of a finite polyhedron K of dimension less than q into X . It follows that

$$f^*: H^q(X; Q) \rightarrow H^q(S^q; Q)$$

is trivial, and since $u_i \cup v_j = 0$, the functional cup-products

$$u_i \cup_f v_j \in H^q(S^q; Q)$$

are defined whenever u_i, v_j are defined. It is easy to show from Lemma 2, by making use of the projections $p_{i,j}: X \rightarrow A_i \vee B_j$, that

$$\begin{aligned} u_i \cup_f v_j &= 0 && \text{if } i \neq j, \\ &\neq 0 && \text{if } i = j. \end{aligned}$$

We have supposed that $u_i \cup_f v_i$ has been defined for an infinite number of values of i .

Suppose f is factored into $g \circ h$, where $g: K \rightarrow X$. Then, since $\dim(K) < q$, $H^q(K; Q) = 0$, and the functional cup-product \cup_h defines a bilinear pairing

$$H^m(K; Q) \otimes H^n(K; Q) \rightarrow H^q(S^q; Q),$$

such that

$$\begin{aligned} (g^*u_i) \cup_h (g^*v_j) &= u_i \cup_f v_j = 0 && \text{if } i \neq j, \\ &\neq 0 && \text{if } i = j. \end{aligned}$$

This implies that infinitely many of the elements g^*u_i are linearly independent; this is impossible since K is a finite complex. Hence $\omega(\gamma) \neq 0$.

REMARK. Thus far any field could be substituted for the rationals Q . The next step does not work for fields of finite characteristic.

Let p be any nonzero integer. Then

$$p\gamma = \sum_{i \in J} [p\alpha_i, \beta_i];$$

and clearly $\omega_Q(p\alpha_i) = p\omega_Q(\alpha_i)$ is not zero if $\omega_Q(\alpha_i) \neq 0$. Therefore the argument shows that $\omega(p\gamma) \neq 0$, that is, $\omega(\gamma)$ is not of finite order. But the kernel of the coefficient homomorphism $H_q(X) \rightarrow H_q(X; Q)$ is the torsion subgroup of $H_q(X)$. Therefore $\omega(\gamma)$ is not in this kernel, and $\omega_Q(\gamma) \neq 0$. This completes the proof.

3. Proof of Theorem 1. Let $A_n = B_n = \bigvee_{i \in J} S_i^n$, and let $X = \bigvee_{n \in J} (A_n \vee B_n)$. Then X is homeomorphic to A_n ; hence if the group $\omega_Q(\pi_p(X))$ is not zero it cannot be countable, for it must contain the direct product $\prod_{n \in J} \omega_Q(\pi_p(A_n))$.

We now construct, by induction on t , elements

$$\gamma_t \in \pi_{t(r-1)+1}(X, *)$$

such that $\omega_Q(\gamma_t) \neq 0$ in $H_{t(r-1)+1}(X; Q)$. This is certainly possible for

$t=1$. Therefore there are elements $\alpha_n \in \pi_r(A_n, *)$ with $\omega_Q(\alpha_n) \neq 0$. Suppose that $\gamma_1, \dots, \gamma_{t-1}$ have been constructed. Corresponding to γ_{t-1} there are elements

$$\beta_{n,t-1} \in \pi_{(t-1)(r-1)+1}(B_n, *)$$

with $\omega_Q(\beta_{n,t-1}) \neq 0$. According to Theorem 2 the element

$$\gamma_t = \sum_{n \in J} [\alpha_n, \beta_{n,t-1}] \in \pi_{t(r-1)+1}(X, *)$$

satisfies $\omega_Q(\gamma_t) \neq 0$. This completes the induction and the proof of Theorem 1.

By a similar means we may prove

THEOREM 3. *Let A_n ($n \in J$) be simply-connected spaces such that for some $r > 1$*

$$0 \neq \omega_Q(\pi_r(A_n, *)) \subset H_r(A_n; Q).$$

Then $H_q(\bigvee_{n \in J} A_n, Q)$ is not countable for $q \equiv 1 \pmod{r-1}$, $q > 1$.

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