A NOTE ON HOMOGENEOUS COMPLEX CONTACT MANIFOLDS

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1. Introduction. A complex contact manifold is by definition a complex analytic manifold of odd dimension[2] $2n+1$ over which is defined to within an analytic scalar multiple an analytic pfaffian form ω of class $2n+1$, i.e. such that $\omega\wedge d\omega^n\neq 0$ at any point. Such a manifold is given [4] by a covering by coordinate neighborhoods $\{U_i\}$ on each of which is defined a form ω_i of the type described with $\omega_i=f_{ij}\omega_j$ whenever $U_i\cap U_j\neq\emptyset$, f_{ij} being nonvanishing, analytic scalar functions. A natural question arises as to whether the contact form may be chosen globally, that is the ω_i so chosen that each $f_{ij}\equiv 1$. It was shown by S. Kobayashi [4] that this can be done if and only if the first Chern class, $c_1(M)$, vanishes. Here and in what follows we will suppose M to be compact. Under the additional assumption that M is simply connected and homogeneous both with respect to the complex and the contact structure the author [2] characterized and enumerated those manifolds for which $c_1(M)\neq 0$. In what follows it is shown that these additional assumptions already imply $c_1(M)\neq 0$, so that combining this with the results of [2] we obtain:

THEOREM I. *Let M be a compact, simply connected, homogeneous complex contact manifold containing more than one point. Then $c_1(M)\neq 0$, hence ω can not be globally defined; M is Kählerian; and there is exactly one such manifold corresponding to each of the classes of simple Lie groups A_n, B_n, C_n, and D_n and the five exceptional simple groups. No other manifolds satisfying these hypotheses exist.*

When we make no homogeneity assumption, the classical example of such a manifold is the bundle of complex "co-directions" over a complex analytic manifold of dimension $n+1$: the fibres are complex projective spaces of dimension n, and with local coordinates $z^1, \cdots, z^{n+1}$ in the base space, $p_1, \cdots, p_{n+1}$ (homogeneous) coordinates in the fibre we have $\omega=p_1dz^1+\cdots+p_{n+1}dz^{n+1}$. In this connection the question arises as to whether the spaces of Theorem I are already included in this example. This question is answered by the following:

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[2] Throughout dimension refers to complex dimension and analytic means complex analytic.

THEOREM II. *With the exception of the manifolds corresponding to* A_n, *none of the manifolds of Theorem* I *can be homeomorphic to a bundle of complex "co-directions" over a complex manifold, i.e. to an example of the type described above.*

2. **Proof of Theorem I.** Following H. C. Wang [5], we will refer to a compact, simply connected, homogeneous complex manifold as a C-space. The notation used in this section is that of [5] and [2] and for brevity will not be redefined. Suppose G/L to be a C-space with a globally defined contact form ω which is G-invariant. Without loss of generality we suppose G to be complex semi-simple. By proceeding as in the real case [3, §4] we will arrive at a contradiction which shows the impossibility of a *globally* defined contact form. This together with the results of [2] gives Theorem I. Let $K=\{g\in G \mid \mathrm{ad}(g)^*\omega^*=\omega^*\}$ where $\mathrm{ad}(g)^*$ denotes the transformation on forms on the Lie algebra $\hat{G}$ induced by the inner automorphism $\mathrm{ad}(g): G\to G$ and $\omega^*=p^*\omega$, $p: G\to G/L$ being the natural map. Clearly K is a closed subgroup and contains L. As in the real case we see $\hat{K}=\{X\in\hat{G} \mid d\omega^*(X, \hat{G})=0\}$ and therefore $\hat{K}$ is a complex Lie algebra; hence K is a closed, complex subgroup and G/K a homogeneous complex manifold. There is a natural analytic map of G/L onto G/K so we know G/K to be compact and connected. If K_0 is the identity component of K, then the same statements apply to K_0 and G/K_0 provided L is connected, which is, in fact, the case since G is connected and G/L is simply connected. Then from the homotopy sequence of $G/L\to G/K_0$ it follows that G/K_0 is simply connected and hence is a C-space. It contains more than a single point for, as in the real case, $\omega\wedge d\omega^n\neq 0$ implies $\dim \hat{K}=1+\dim \hat{L}$ and, if G/K_0 is a single point, then $K_0=G$ and K_0/L is an abelian group of complex dimension one. Since it is compact and simply connected, it too contains but one point which is contrary to assumption.

Now let $Z\in\hat{G}$ correspond to ω^* relative to the Killing form (X, Y) on $\hat{G}$. Then $\omega^*(X)=(Z, X)$ and $d\omega^*(X, Y)=\omega([X, Y])/2$. Thus $d\omega^*(X, Y)=0$ if and only if $([Z, X], Y)=(Z, [X, Y])=0$. Since the Killing form is nondegenerate, the first of these vanishes for all $Y\in\hat{G}$ if and only if $[X, Z]=0$, therefore $\hat{K}=\hat{C}(Z)$, the centralizer of Z, i.e. $\hat{C}(Z)=\{X \mid [X, Z]=0\}$. The desired contradiction is then obtained from the following:

LEMMA. *Let G be a connected, complex, semi-simple Lie group and K a closed, connected, complex subgroup whose Lie algebra $\hat{K}$ is the centralizer of some $Z\in\hat{G}$. Then, if G/K is a C-space, it contains only one point.*

PROOF. According to Wang [5], if G/K is a C-space, then for the

Lie algebras we have the following situation. There exists a Cartan subalgebra $\hat{H}\subset\hat{G}$ with rational basis $h_1, \cdots, h_b, h_{b+1}, \cdots, h_a, h_{a+1}, \cdots, h_l$, with $b=2u$, where $\hat{K}$ has as basis $k_1, \cdots, k_u, h_{b+1}, \cdots, h_a, h_{a+1}, \cdots, h_l; E_{\pm\alpha_1}, \cdots, E_{\pm\alpha_s}, E_{\sigma_1}, \cdots, E_{\sigma_r}$ where $k_1=ih_1+ch_2$, $k_2=ih_3+ch_4, \cdots, k_u=ih_{b-1}+ch_b$, c a real number, and $\alpha_1, \cdots, \alpha_s$, $\sigma_1, \cdots, \sigma_r$ are all positive roots and $\pm\alpha_1, \cdots, \pm\alpha_s$ are all those roots which vanish on each of the vectors $h_1, \cdots, h_a$. The maximal semi-simple subalgebra $\hat{Q}$ of $\hat{K}$ is spanned by $h_{a+1}, \cdots, h_l$ and $E_{\pm\alpha_1}, \cdots, E_{\pm\alpha_s}$. Since $Z\in\hat{K}$, we may write with suitable $h\in\hat{H}\cap\hat{K}$, $Z=h+\sum_{i=1}^s a_iE_{\alpha_i}+\sum_{i=1}^s b_iE_{-\alpha_i}+\sum_{j=1}^r d_jE_{\sigma_j}$. We shall prove $Z=0$, whence $\hat{G}=\hat{K}$ and G/K is a single point. If $X\in\hat{K}$ is either in $\hat{H}$ or a root vector E_τ, then since $[X, Z]=0$, each summand of $[X, h]+\sum_{i=1}^s a_i[X, E_{\alpha_i}]+\sum_{i=1}^s b_i[X, E_{-\alpha_i}]+\sum_{j=1}^r d_j[X, E_{\sigma_j}]$ vanishes since each lies in a different summand of a direct sum decomposition of $\hat{G}$. Thus $[E_\sigma, h]=\sigma(h)E_\sigma$ vanishes for each positive root σ, hence $h=0$. Multiplying by $E_{+\alpha}$ or $E_{-\alpha}$ we obtain $a_i=0=b_i$ for each i. Finally let σ_{j_0} be one of the roots $\sigma_1, \cdots, \sigma_r$ and k, $1\leqq k\leqq a$, be such that $\sigma_{j_0}(h_k)\neq 0$. There must be at least one such or σ_{j_0} would be among the $\pm\alpha_i$. If $b+1\leqq k\leqq a$, then, since $h_k\in\hat{K}$ and thus $[h_k, Z]=0$, we at once get $d_{j_0}[h_k, E_{\sigma_{j_0}}]=\sigma_{j_0}(h_k)d_{j_0}E_{\sigma_{j_0}}=0$ and thus $d_{j_0}=0$. But, if $1\leqq k\leqq b$, then either $[ih_k+ch_{k+1}, Z]d_{j_0}=0$ or $[ih_{k-1}+ch_k, Z]d_{j_0}=0$ according to the parity of k. In either case $d_{j_0}=0$ unless the term in brackets vanishes. But this cannot happen since, say in the first case, vanishing of the term in brackets would imply $i\sigma_{j_0}(h_k)+c\sigma_{j_0}(h_{k+1})=0$ and thus since σ_{j_0} is real on h_k and h_{k+1}, $\sigma_{j_0}(h_k)=0=\sigma_{j_0}(h_{k+1})$, contrary to our choice of h_k. A corresponding argument applies to the other case and to each of the σ_j, $j=1, \cdots, r$. Thus each $d_j=0$ and so $Z=0$ as was claimed.

3. **Proof of Theorem II.** We now consider the possibility that the homogeneous complex contact manifold $M=G/L$ might be a bundle of complex co-directions over a complex analytic manifold B of dimension $n+1$. In this case the fibre F would be a complex projective space of dimension n. By Theorem I M is a Kähler manifold, F is also Kählerian, and by Blanchard [1, Prop. II.2, p. 184] we see that B is Kählerian. Concerning the topology of M, F, and B we use the fact that $H^1(F)=0$, which implies that the transgression $H^1(F)\rightarrow H^2(B)$ vanishes, together with the fact that $\pi_1(B)=0$, which follows from the fact that F and M are both simply connected spaces and, of course, implies that $\pi_1(B)$ acts trivially on the cohomology groups of F. This allows us to use Blanchard, op. cit., Theorem II 1.2, p. 178, according to which the real cohomology of M is isomorphic

to that of $B \times F$. In particular for the Poincaré polynomials of M, F, and B we have

$$P_M(t) = P_F(t)P_B(t) = (1 + t^2 + \cdots + t^n)(1 + at^2 + \cdots + t^{n+1})$$

where $a \geqq 1$ since B is Kählerian. It follows that the second Betti number of M, $p_2(M) \geqq 2$. Now let $M = G/L$, G complex semi-simple as before. It is known that $\pi_2(G) = 0$. The homotopy sequence of the bundle $G \to G/L$ yields the exact sequence:

$$0 \to \pi_2(G/L) \to \pi_1(L) \to \pi_1(G) \to 0,$$

which, in particular, says that the rank of $\pi_2(G/L)$ is no greater than the rank of $\pi_1(L)$. Since G/L is simply connected, $\pi_2(G/L) \approx H^2(G/L)$; thus $p_2(M) \leqq \text{rank } \pi_1(L)$. Now examining the individual cases in Theorem I and replacing G and L by their maximal compact subgroups G', L', cf. [2], shows that $L' = A \times T^r$ where A is semi-simple and thus rank $\pi_1(L) = r$. Since $2 \leqq p_2(M) \leqq r$ we must have $r \geqq 2$, but this occurs only in the case corresponding to A_n where $G' = SU(n+1)$ and $L' = SU(n-1) \times T^2$.

4. **Remarks.** I. Since every *homogeneous* complex contact manifold, at least if it is simply connected, is a Kähler manifold, one might ask if this is true in general. However, from Blanchard, op. cit., we see that if B is a non-Kählerian, compact, complex analytic manifold then no bundle over it can be Kählerian. Thus the bundle of co-directions furnishes an example of a non-Kählerian compact complex contact manifold. It will be simply connected if B is chosen simply connected.

II. In the one exceptional case of Theorem II we have $SU(n+1) \supset SU(n) \times T^1 \supset SU(n-1) \times T^2$ so that denoting $SU(n+1)/SU(n-1) \times T^2$ by M, $SU(n+1)/SU(n) \times T^1$ by B and $SU(n) \times T^1/SU(n-1) \times T^2$ by F we see that M is a bundle over B whose fibre F is a projective space $P_{n-1}(C)$ of dimension $n-1$ and whose base B is a projective space $P_n(C)$ of dimension n. And in fact the bundle is homeomorphic to the bundle of tangent co-directions to n dimensional complex projective space.

A proof of this last statement may be sketched as follows. We regard $P_n(C)$ as the space of directions through the origin in C_{n+1}, $n+1$ dimensional Hermitian space, and the complex n-plane orthogonal to a given direction as the tangent space to the corresponding point of $P_n(C)$. Then $SU(n+1)$ is transitive on the pairs (x, d) $=$(point x, tangent direction at x) of $P_n(C)$. The subgroups fixing respectively x and (x, d) are $SU(n) \times T^1$ and $SU(n-1) \times T^2$. Then

the result follows easily if we note that the space of tangent co-directions to $P_n(C)$ is homeomorphic to the space of tangent directions as we may see by choosing a Hermitian metric on $P_n(C)$.

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