A MULTI-POINT GENERALIZATION OF NEWTON’S DIVIDED DIFFERENCE FORMULA

HERBERT E. SALZER

Newton’s divided difference formula for the polynomial of degree \( n - 1 \) determined by the values of the function \( f(x) \) and its divided differences up to the \( (n - 1) \)th order originating from any single point, has the following multi-point generalization:

**Theorem I.** The unique interpolation polynomial \( P(x) \), of degree \( \sum_{i=1}^{r} r_i - 1 \), which is determined by \( f(x_i) = f_i = D^0 f_i \) and the first \( r_i - 1 \) divided differences, \( D^r f_i = [x_{i_1} x_{i_2} \cdots x_{i_m}] \), \( m = 1, \ldots, r_i - 1 \), originating from the \( n \) points \( x_i, i = 1, \ldots, n \), is expressible as

\[
P(x) = \sum_{i=1}^{n} \Pi_i(x)\psi_i(x), \quad \text{where}
\]

\[
\Pi_i(x) = \prod_{j=1, j \neq i}^{n} (x - x_j)(x - x_{i_1}) \cdots (x - x_{i_{r_i}-1}),
\]

\[
\psi_i(x) = \sum_{m=0}^{r_i-1} [x_{i_1} x_{i_2} \cdots x_{i_m}] \psi_{i,m}(x), \quad \text{and}
\]

\[
\psi_{i,m}(x) = \sum_{s=0}^{r_i-m-1} [x_{i_m} x_{i_{m+1}} \cdots x_{i_{m+s}}] 1/\Pi_i(x - x_i)(x - x_{i_1}) \cdots (x - x_{i_{r_i}-1}).
\]

In (3) and (4), \( x_{iq} = x_i \); in (3), \( [x_i] = f_i = D^0 f_i \); in (4), \( [x_{i_m}]_{1/\Pi_i} = 1/\Pi_i(x_{i_m}), \) the \( st \) order divided difference \( [x_{i_m} x_{i_{m+1}} \cdots x_{i_{m+s}}]_{1/\Pi_i} \) is for the function \( 1/\Pi_i(x), \) and \( \psi_{i,s}(x) \) begins \( [x_i]_{1/\Pi_i} + \cdots. \)

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Theorem I is a generalization of the author's previous result involving advancing differences of \( f(x) \), at equal intervals, at each \( x_i \) \([1]\). Also, Theorem I extends the result in \([1]\) beyond generalization to irregular distribution of points \((x_i, x_{i+1}, \ldots, x_{i+s})\) in that the coefficients of \((x-x_i)(x-x_{i+1})\cdots(x-x_{i+m+s+1})\) in \((4)\) are given explicitly as \[x_i x_{i+1} \cdots x_{i+m+s+1}\] whereas in \([1]\) corresponding quantities that are proportional to those coefficients are given implicitly as solutions of a triangular linear system.

Theorem I, which may be regarded as a partitioned form of Newton's divided difference formula, might be applied advantageously, say in reducing the computing error by employing divided differences of lower orders, or where the points \((x_i, x_{i+1}, \ldots, x_{i+s})\) are at the same irregular intervals, so that the divided differences \(x_i x_{i+1} \cdots x_{i+m+s+1}\) are the same linear combinations of \(1/\Pi_i(x_{i+m+s+1})\), \(i=0, 1, \ldots, s\), for every \(i\).

**Proof of Theorem I.** We use the following formula \([2]\) for the \(m\)th divided difference of any \(F(x)\) for the \(m+1\) points \(x_0, x_1, \ldots, x_m\):

\[
\frac{D^nF}{x=x_i} = \sum_{i=0}^{m} \left\{ \frac{F(x_i)}{\prod_{j=0, j \neq i}^{m} (x_i - x_j)} \right\}.
\]

(5) \( D^mF \equiv [x_0 x_1 \cdots x_m]_F = \sum_{i=0}^{m} \left\{ \frac{F(x_i)}{\prod_{j=0, j \neq i}^{m} (x_i - x_j)} \right\}. \)

To show \(D^mP(x)|_{x=x_i} = [x_i x_{i+1} \cdots x_m]_P\), \(m=0, 1, \ldots, r_i-1\), we note first from (5) and (2) that the divided difference operator \(D^m \cdots |_{x=x_i}\) annihilates every term in (1) except the \(i\)th. It suffices to show that

(6) \( D^{m'} \Pi_i(x) \psi_{i,m}(x) |_{x=x_i} = \delta_{m'}^{m'}, \quad m' = 0, 1, \ldots, r_i - 1. \)

For \(m' \leq m\), the result follows from (4) and (5), for arbitrary values of the coefficients of \((x-x_i)(x-x_{i+1})\cdots(x-x_{i+m+s+1})\) in (4) beyond the first.

We prove (6) for any \(m' > m\) by induction, assuming (4) as far as \(x_i x_{i+1} \cdots x_{i+m+s-1} / \Pi_i(x-x_i)(x-x_{i+1})\cdots(x-x_{i+m+s-2})\) to be equivalent to (6) for \(m' \leq m+s-1\), which was just shown for \(s=1\). For (6) to hold when \(m' = m+s\), denoting in (4) the coefficient of \((x-x_i)(x-x_{i+1})\cdots(x-x_{i+m+s-1})\) by \(a_{i,m}^{(s)}\), application of (5) to \(x_i x_{i+1} \cdots x_{i+m+s+1}\) \[\Pi_i(x_{i+s})/(x_i - x_{i+s}) \]

\[+ \Pi_i(x_{i+s+1})/(x_{i+s+1} - x_m)(x_{i+s+1} - x_{i+s+2}) \cdots (x_{i+m+s+1} - x_{i+m+s+2}) + \cdots \]

\[+ \Pi_i(x_{i+s})/(x_{i+s} - x_m) \cdots (x_{i+m+s+1} - x_{i+m+s+2}) \]

\[= (1/\Pi_i(x_{i+s})) \left\{ \left[ \sum_{i=0}^{m} \left\{ \frac{\Pi_i(x_{i+s})}{(x_{i+s} - x_m)} \right\} \right] \right\} \]

(7) \[+ \cdots + [x_i x_{i+1} \cdots x_{i+m+s+1}]/\Pi_i \left\{ \Pi_i(x_{i+s})/(x_{i+s} - x_m) \right\} + a_{i,m}^{(s)} \Pi_i(x_{i+s}). \]
The quantities in braces in (7) are
\[ [x_{i_1} \cdots x_{i_m}]_{\Pi_i}, \ldots, [x_{i_m+1} x_{i_m+2}]_{\Pi_i}. \]

We now employ the Popoviciu-Steffensen theorem for the \( m \)th
divided difference of the product \( f(x)\phi(x) \) in terms of the divided
differences of \( f(x) \) and \( \phi(x) \) \([3; 4]\):\(^1\)

\[
(8) \quad [x_0 x_1 \cdots x_m]_{f\phi} = \sum_{i=0}^{m} [x_0 \cdots x_i]_f [x_i \cdots x_m]_{\phi}.
\]

Applying (8) to the product \( 1 = \{1/\Pi_i(x)\} \Pi_i(x) \) for the points
\( (x_{i_1}, x_{i_1+1}, \ldots, x_{i_m+1}) \), we have

\[
0 = (1/\Pi_i(x_{i_1})) [x_{i_1} \cdots x_{i_m+1}]_{\Pi_i} + \cdots
\]

\[
+ [x_{i_1} \cdots x_{i_m+1}]_{1/\Pi_i} [x_{i_m+2} x_{i_m+3}]_{\Pi_i} + \cdots
\]

\[
+ [x_{i_1} \cdots x_{i_m+1}]_{1/\Pi_i} [x_{i_m+1}]_{\Pi_i}.
\]

Comparison of (9) with (7) shows \( a_{i_0}^{(m)} = [x_{i_1} \cdots x_{i_m+1}]_{1/\Pi_i} \).

The uniqueness of \( P(x) \) is apparent, since specification by \( f_i, \)
\( \Delta f_i, \cdots, \Delta^{n-1} f_i, i = 1, \ldots, n, \) is equivalent to specification by certain
\( \sum_{i=1}^{n} r_i \) functional values \( f_i, f_{i_1}, \cdots, f_{i_r-1} \) at \( x_i, x_{i_1}, \cdots, x_{i_r-1} \),
for which the Lagrange interpolation polynomial of degree \( \sum_{i=1}^{n} r_i - 1 \)
(a rearrangement of \( P(x) \)) is unique.

References

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2. L. M. Milne-Thomson, The calculus of finite differences, Macmillan, London,
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3. T. Popoviciu, Sur quelques propriétés des fonctions d'une ou de deux variables
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\(^1\) Popoviciu published a statement of (8) without proof in his thesis \([3]\), apparently
unknown to Steffensen whose proof appeared 5 or 6 years later \([4]\).