ON 3-MANIFOLDS THAT ARE NOT SIMPLY CONNECTED

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Let X be a connected 3-manifold. Then X is simply connected if and only if each simple closed curve in X lies in a simply connected neighborhood. On the other hand, each point of X trivially lies in a simply connected neighborhood with or without X itself simply connected. However, we will show that arcs are big enough to reflect the global property of X as to simple connectedness in the sense that X is simply connected if and only if each arc in X lies in a simply connected neighborhood.

THEOREM. *Let X be a 3-manifold that is not simply connected. Then there exists an arc in X that does not lie in any simply connected neighborhood.*

PROOF. There exists a simple closed polygon L that is not deformable to a point over X, i.e., the inclusion map of L into X is essential. To see this, we choose a loop representing a nontrivial element of the fundamental group of X and apply thereto the simplicial approximation theorem followed by obvious modifications if necessary. Let T be a polyhedral solid torus which is a closed tubular neighborhood of L. Evidently, T does not lie in a simply connected neighborhood.

We now proceed to construct a Cantor set of Antoine type [1]. Let $T_1, T_2, \cdots, T_k$ be a finite number of polyhedral solid tori of diameter less than $1/2$ that are situated in Int T as suggested by Figure 1. We show that the sum T^1 of the T_i does not lie in a simply connected neighborhood. Suppose it does.

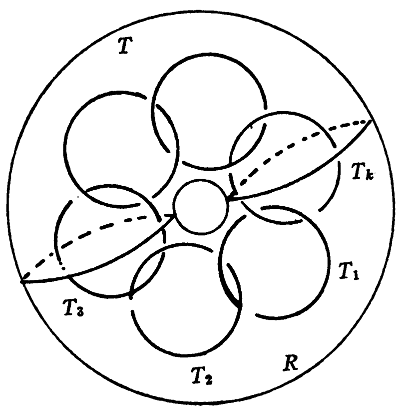

FIGURE 1

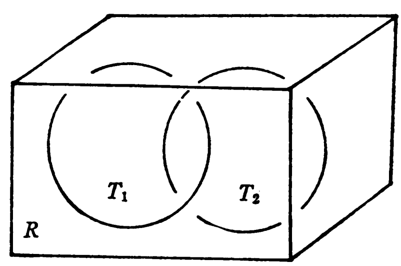

FIGURE 2

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Let U be a simply connected open set containing T^1. We cut a polyhedral cube R containing T_1 and T_2 out of T as suggested by Figures 1 and 2. T_1 and T_2 are linked in R in the sense that neither of the two can be deformed to a point over the complement in R of the other. We show that they can be joined by an arc in Int $R\cdot U$. Let L_i be simple closed polygon that circle on Bd T_i longitudinally once and x_i be points on L_i, $i=1$, 2. Since U is simply connected, there exists a singular disk D bounded by L_1, $D\subset U$. We may suppose that D meets Bd R at a 1-dimensional set. Let D' be the component of D. Int R that contains L_1. Then $D'+$Bd R contains a singular disk D'' bounded by L_1. Since D'' meets L_2 and L_2 does not meet Bd R, the existence of the required arc is established.

Let C_1 be an arc in Int $R\cdot U$ that joins x_1 and x_2. Similarly, we construct arcs C_i joining x_i and x_{i+1}, $x_i\in T_i$, subscripts being taken modulo k, such that $\sum C_i$ is a simple closed curve in U which is deformable to L in Int T. Since U is simply connected, L is deformable to a point over X, contradictory to the choice of L. Thus T^1 does not lie in a simply connected neighborhood.

We now construct polyhedral solid tori T_{i1}, T_{i2}, $\cdots$ in Int T_i such that each T_{ij} is of diameter less than 1/4, T_{ij} are situated in Int T_i as T_j are in Int T. Let T^2 be the sum of T_{ij}. Then T^2 does not lie in a simply connected neighborhood. For if it did, then the simply connected neighborhood must contain for each i a simple closed polygon in Int T_i that is deformable within T_i to a simple closed polygon that lies on Bd T_i and circles T_i longitudinally once, and which in turn would imply that it contained a curve which is deformable to L in X, thus the same contradiction.

Similarly, we construct sets T^3, T^4, $\cdots$ such that no T^i lies in a simply connected neighborhood, each T^i contains T^{i+1} in its interior and the intersection of T^i is a Cantor set C. There exists in Int T an arc P containing C. Then P does not lie in any simply connected neighborhood in X.

We wish to thank the referee for his suggestion which made the proof considerably shorter.

Reference

1. L. Antoine, *Sur l'homéomorphie de deux figures et de leur voisinages*, J. Math. Pures Appl. 86 (1921), 221–325.

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