INJECTIVE MORPHISMS OF REAL ALGEBRAIC VARIETIES

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1. A recent note of D. J. Newman [3] shows that for polynomial maps of the real plane into itself injectivity implies surjectivity. The present note combines two independently obtained corrections and generalizations of Newman's proof.

We need some preliminaries. Take the complex numbers as universal domain and define a *real algebraic set* to be the set of real points of an algebraic set that is defined over the reals. A real algebraic set V is Zariski-dense in its Zariski closure $\overline{V}$, which is an algebraic set defined over the reals, and V is the set of real points of $\overline{V}$. Any algebraic subset of $\overline{V}$ meets V in a real algebraic subset. V is *irreducible* (in its Zariski topology), i.e. V is a *real algebraic variety*, if and only if $\overline{V}$ is irreducible (over the *complex* numbers), we define $\dim V = \dim \overline{V}$, and we call $P \in V$ *simple* if P is a simple point of $\overline{V}$; such points P exist, and there exist uniformizing parameters that are defined over the reals for $\overline{V}$ at such a point P, hence real local power series expansions, so that at each of its simple points V, in its ordinary topology, is locally a real analytic manifold of dimension $\dim V$. For real algebraic sets V, W define a *rational map* f: $V \rightarrow W$ to be the restriction to $V \times W$ of a rational map $\bar{f}$: $\overline{V} \rightarrow \overline{W}$ that is defined over the reals; the rational map f: $V \rightarrow W$ is a *morphism* if $\bar{f}$ is defined at each point of V. Supposing the morphism of real algebraic varieties f: $V \rightarrow W$ to be such that $f(V)$ is Zariski-dense in W, a simple point $P \in V$ may be found such that $f(P)$ is simple on W and df has the correct rank $\dim V - \dim W$ at P, implying that, for the real analytic structure of V, W at $P, f(P)$ respectively, f is locally a projection onto a direct factor; in particular, if f is finite-to-one, then $\dim V = \dim W$ and $\bar{f}$ has a finite degree N.

Specific examples of real algebraic sets: The real part of a complete nonsingular algebraic curve that is defined over the reals is the disjoint union of a finite number of circles, each with a real analytic structure. If V is a real algebraic set of dimension one then a Zariski-open subset of V is isomorphic to a Zariski-open subset of a nonsingular projective model of V over the reals, hence the ordinary topology of V is obtained by taking the topological sum of a finite number of points and circles, effecting a finite number of identifications, and deleting a finite number of points. If V is a real hypersurface in $\boldsymbol{R}^n$,

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then V is the zero locus of a single real polynomial in n variables, say $f \in \boldsymbol{R}[X_1, \cdots, X_n]$; since V has simple points and rank $df = 1$ at these, f takes on both positive and negative values in $\boldsymbol{R}^n$—thus V separates $\boldsymbol{R}^n$, in the ordinary topology.

2. The following is the main point of Newman's proof.

If f: $V \rightarrow W$ is an injective morphism of real algebraic sets and $f(V)$ is Zariski-dense in W, then dim $V =$ dim W *and $f(V)$ contains a Zariski-dense Zariski-open subset of W.*

To prove this, we may take V, W irreducible, in which case we already know that dim $V =$ dim W and that $\bar{f}$ has finite degree N. Thus there exists a Zariski-closed proper subset $F \subset \overline{W}$ such that $\bar{f}^{-1}(Q)$ consists of precisely N points whenever $Q \in \overline{W} - F$. For Q real, $\bar{f}^{-1}(Q)$ consists of real points plus pairs of complex conjugate points, so taking $Q \in f(V) - F$ we deduce that N is odd and taking $Q \in W - F$ that $\bar{f}^{-1}(Q)$ has at least one real point. Thus $f^{-1}(Q)$ is not empty if $Q \in W - W \cap F$.

Note that if V, W are real algebraic varieties and the above conditions hold, then $\bar{f}$ has odd degree. Conversely, if we delete the assumption of injectiveness and merely assume the degree of $\bar{f}$ odd, then the same conclusion follows.

3. We now prove the generalization of Newman's result.

If f: $\boldsymbol{R}^n \rightarrow \boldsymbol{R}^n$ is an injective morphism (of real algebraic sets, e.g. a real polynomial map), *then f is also surjective.*

First, injectivity and continuity in the ordinary topology imply that f is a homeomorphism between $\boldsymbol{R}^n$ and $\boldsymbol{R}^n - X$, where $X \subset \boldsymbol{R}^n$ is closed in the ordinary topology (invariance of domain). Consider the real algebraic sets $V_1 = \overline{X} \cap \boldsymbol{R}^n$, $V_2 = f^{-1}(V_1)$. f is a homeomorphism (in the ordinary topology) between $\boldsymbol{R}^n - V_2$ and $\boldsymbol{R}^n - V_1$. If $V_3 \subset V_1$ is the Zariski-closure of $f(V_2)$, the previous section tells us that dim V_3 $=$ dim V_2 and that $f(V_2)$ contains a Zariski-dense Zariski-open subset of V_3; in particular, $V_3 - f(V_2)$ is nowhere dense in V_3. Since $X = V_1 - f(V_2) = (V_1 - V_3) \cup (V_3 - f(V_2))$ and X is Zariski-dense in V_1, we must have $V_1 - V_3$ Zariski-dense in V_1, so that dim $V_3 <$ dim V_1 unless V_1 is empty (which is to be proved). We are reduced to proving the following: *if V_1, V_2 are* (*closed*) *real algebraic subsets of $\boldsymbol{R}^n$, then $\boldsymbol{R}^n - V_1$ and $\boldsymbol{R}^n - V_2$ are homeomorphic* (*in the ordinary topology*) *only if* dim $V_1 =$ dim V_2.

In proving the above statement we may assume dim V_1, dim $V_2 < n$. We wish first to remark that a quite elementary proof can be given for the case $n \leqq 3$. As a matter of fact, we need only the following statements (each of which is either obvious or a direct consequence of the last paragraph of §1): If V is a proper real algebraic subset of $\boldsymbol{R}^n$, where $n \leqq 3$, then

(a) $\boldsymbol{R}^n$ is contractible,

(b) $\boldsymbol{R}^n - V$ is connected if and only if dim $V < n-1$,

(c) if $n=2$ and dim $V=0$ then $\boldsymbol{R}^n - V$ is not simply connected,

(d) if $n=3$ and dim $V=0$ then $\boldsymbol{R}^n - V$ is simply connected, but contains a 2-sphere not homotopic to a point,

(e) if $n=3$ and dim $V=1$ then the one-point compactification of $\boldsymbol{R}^n$ contains that of V, which contains a topological circle, so $\boldsymbol{R}^n - V$ is not simply connected.

To complete the proof in general we use the homology theory of Borel and Haefliger [1], with integers modulo two as coefficients. To each locally compact topological space X are associated $\boldsymbol{Z}_2$-modules $H_i(X) = H_i(X;\ \boldsymbol{Z}_2)$, one for each integer i, vanishing for $i >$ (topological dimension of X), reducing to the ordinary simplicial homology groups if X is a finite simplicial complex and to the usual relative homology groups if X is the complement of a subcomplex of a finite simplicial complex. There is an exact sequence relating the homologies of a space, a closed subspace and its complement, and, most essential, if X is a real algebraic set of dimension m there exists a (unique) fundamental class, i.e. an element of $H_m(X)$ which induces at each simple point of X the generator of the local mth homology group. This being so, consider for any proper real algebraic subset V of $\boldsymbol{R}^n$ the exact sequence

$$\cdots \to H_i(V) \to H_i(\boldsymbol{R}^n) \to H_i(\boldsymbol{R}^n - V) \to H_{i-1}(V) \to \cdots.$$

We know that $H_i(\boldsymbol{R}^n)$ is $\boldsymbol{Z}_2$ or 0, depending on whether $i=n$ or $i \neq n$, and $H_i(V)=0$ if $i > \dim V$, $H_{\dim V}(V) \neq 0$. Hence

$$\begin{aligned} H_n(\boldsymbol{R}^n - V) &\approx \boldsymbol{Z}_2 \oplus H_{n-1}(V), \\ H_i(\boldsymbol{R}^n - V) &\approx H_{i-1}(V) \qquad \text{if } i < n. \end{aligned}$$

Thus the knowledge of the groups $\{H_i(\boldsymbol{R}^n - V)\}$ determines dim V, and we are done.

4. Here are some more, simpler, consequences of §2.

If f: $V \to W$ is an injective morphism of complete nonsingular irreducible real algebraic curves, then f is bijective. For the proof of the result of §2 gives precise information in this case, since, taking $\overline{V}$ complete and nonsingular, as we may, over each point of W lies the same number of points of $\overline{V}$, counting multiplicities.

If f: $V \to V$ is an injective morphism of the real algebraic curve V into itself, then f is bijective. This follows from the following more general topological fact: If the topological space V is the complement of a finite subset of a finite one-complex and if $f\colon V\to V$ is a continuous injection such that $V-f(V)$ consists of a finite number of points, then f is a bijection. For the proof, note that the vertices of V (i.e., points adherent to more than two ends of open line segments) are mapped by f into vertices. Since the vertices are finite in number, f is bijective on the vertices, and we may delete these from V. V is now the topological sum of a finite number of disjoint points, line segments (open, half-open, or closed), and circles, and the proof here is straightforward.

If $f\colon G\to H$ is an injective homomorphism of real algebraic groups then $f(G)$ is a real algebraic subgroup of H. Using the Zariski topology, this reduces to showing that *a dense abstract subgroup of a real algebraic group which contains a nonempty open subset is the whole group*, which is a consequence of the fact that the closure of its complement is a proper closed subset invariant under a dense set of translations, therefore invariant under all translations, therefore empty.

5. We remark finally the following easy proof that *if $f\colon k^n\to k^n$ is an injective polynomial map, where k is any algebraically closed field, then f is also surjective*: Here f must be birational or purely inseparable and a result of Chevalley [2, p. 195] shows that f is open, so $f(k^n)=k^n-X$, with X closed. Whether or not f^{-1} is defined at a particular point depends on the poles of rational functions, so either dim $X=n-1$ or X is empty. If X were nonempty a nonconstant polynomial function on k^n with zero locus contained in X would give a similar function with no zero locus, which is impossible.

References

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