A THEOREM OF LEVITZKI

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Levitzki [2] proved that in a ring satisfying the ascending chain condition on left-ideals every left-ideal consisting of nilpotent elements is nilpotent. More recently, Goldie [1] has given a simpler proof. We present here a proof which is, in our opinion, even simpler and more elementary than that of Goldie. Moreover, it is completely self-contained.

THEOREM (LEVITZKI). *In a ring with ascending chain condition on left-ideals a nil left-ideal must be nilpotent.*

PROOF. We claim that we may assume that the ring R has no nonzero nilpotent left-ideals, for if it did, it would have a nonzero nilpotent two-sided ideal, hence a maximal such, N. If the result were false, in R/N, which has no nilpotent left-ideals we would have a nonzero nil left-ideal.

So suppose that R is a ring with ascending chain condition on left-ideals which has no nonzero nilpotent left-ideals and that $A \neq (0)$ is a nil left-ideal of R. Let $a \neq 0 \in A$, and let $\mathfrak{M} = \{ax \neq 0 \mid x \in R\}$. For $ax \in \mathfrak{M}$ let $L(ax) = \{y \in R \mid yax = 0\}$; these give us a set of left-ideals of R, which, by the ascending chain condition has maximal elements. We denote these maximal elements by $L(ax_i)$.

If $t \in R$ and $ax_it \neq 0$ then $L(ax_it) \supset L(ax_i)$ which, together with $ax_it \in \mathfrak{M}$ forces $L(ax_it) = L(ax_i)$. We claim that given any finite number of such maximal ax_i,—$ax_1, \cdots, ax_n$—then there is an element $u \neq 0$ in Rax_1R such that $ax_1u = \cdots = ax_nu = 0$. Suppose such a u has been found such that $ax_1u = \cdots = ax_{n-1}u = 0$. If $ax_nu = 0$ then we are done. So suppose any $u \neq 0 \in Rax_iR$ annihilating $ax_1, \cdots, ax_{n-1}$ does *not* annihilate ax_n. We claim there is such a $u \neq 0$ such that $uax_n \neq 0$; for if $uax_n = 0$ for all $u \in Rax_1R$ which annihilate $ax_1, \cdots, ax_{n-1}$, then since uR also does the trick, $uRax_n = (0)$, whence $(Rax_nu)^2 = (0)$. But then Rax_nu is a nilpotent left-ideal, so must be (0). This forces $ax_nu = 0$ the desired result. Thus we may suppose that $uax_n \neq 0$. Now $(uax_n)^t = 0$ for some t, since $x_nua \in A$ is nilpotent. If $(uax_n)^t = 0$, $(uax_n)^{t-1} \neq 0$, then $uax_n(uax_n)^{t-1} = 0$. Since $u \in L(ax_n(uax_n)^{t-1})$ but $u \notin L(ax_n)$, by the remark at the beginning of this paragraph, $ax_n(uax_n)^{t-1} = 0$. Thus $(uax_n)^{t-1} \neq 0$ annihilates $ax_1, \cdots, ax_{n-1}$ and ax_n; moreover since $u \in Rax_1R$, $(uax_n)^{t-1}$ also is.

Consider the ascending chain of left-ideals Rax_1, $Rax_1 + Rax_2, \cdots,$

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$Rax_1+Rax_2+\cdots+Rax_n, \cdots$. It must terminate; thus for some n, $Rax_i \subset Rax_1+\cdots+Rax_n$ for *all* ax_i such that $L(ax_i)$ is maximal. Now ax_1r is either 0 or a maximal ax_i; whence $Rax_1R\subset Rax_1+\cdots+Rax_n$. But there is a $u\neq 0$ in Rax_1R such that $ax_1u=\cdots=ax_nu=0$. Thus $Rax_1Ru\subset Rax_1u+\cdots+Rax_nu=(0)$. Therefore, since $Ru\neq 0\subset Rax_1R$, $(Ru)^2\subset Rax_1RRu=(0)$. We have produced a nonzero nilpotent left-ideal in R!. This contradiction proves the theorem.

Clearly the proof would work if A were a nil right-ideal, for if $0\neq a\in A$, then Ra would be a nonzero nil left-ideal of R.

References

1. A. W. Goldie, *Semi-prime rings with maximum condition*, Proc. London Math. Soc. **10** (1960), 201–220.

2. Nathan Jacobson, *The structure of rings*, Amer. Math. Soc. Colloq. Publ. Vol. 37, Amer. Math. Soc., Providence, R. I., 1956.

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A REMARK ON DIRECT PRODUCTS OF MODULES

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It is a standard result of abelian group theory that the direct product of an infinite number of infinite cyclic groups is not free [**3**, p. 48]. Various generalizations of this fact to modules over an arbitrary ring are contained in [**1**]. In this note we restrict our attention to integral domains, and in this setting present a generalization of the aforementioned result which is similar to, but in some sense stronger than, those of [**1**]. We then apply our theorem to show that any factor group A of a direct product of infinite cyclic groups possesses the following amusing property: If A is a subgroup of a direct sum of reduced torsion-free abelian groups, then it must be contained in the direct sum of a finite number of them.

Throughout this discussion, R will be an integral domain which is not a field. All R-modules considered will be assumed to be unitary.

Lemma. *Let A be a torsion-free R-module, and set $A'=\bigcap_{\lambda\in R^*}\lambda A$. Then A' is divisible. In particular, if A is reduced (i.e., has no divisible submodules) then $\bigcap_{\lambda\in R^*}\lambda A=0$.*[1]

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[1] R^* means $R-\{0\}$.

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